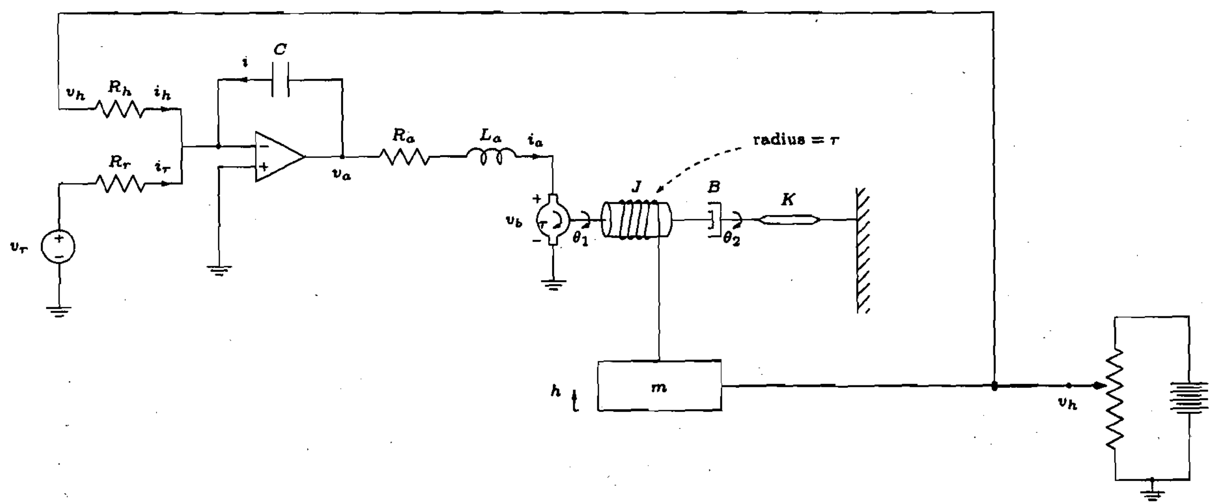
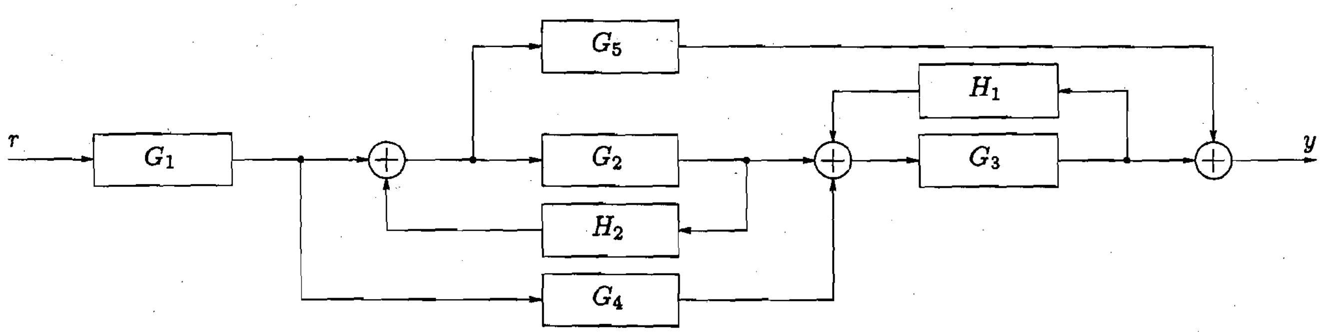
EE 231 | Exam#1 | Mar. 06, 2008

75 minutes

1. In the following system, a motor is used to control the height of a massive structure. The height h is detected as a voltage signal v_h by the use of a variable resistor, such that $v_h = k_h h$. Obtain the detailed block diagram of the system, where v_r is the input and h is the output, and show the variables v_r, i_r, v_h, i_h, i, v_a, i_a, v_b, τ, θ_1, θ_2, and h on the block diagram. (30pts)



2. Consider the following block diagram of a control system.



Determine the transfer function of the system. Show all your work. (25pts)

3. The acceleration measured by a sensing device that is placed along a flexible beam is given by

$$a_m(t) = \begin{cases} -a^2(t)\sin\big((\pi/(2l))x(t)\big), & \text{if } a(t) < 0; \\ a^2(t)\sin\big((\pi/(2l))x(t)\big), & \text{if } a(t) \geq 0; \end{cases}$$

where a and x are the actual acceleration and the displacement along the flexible beam of length l, respectively.

(a) Obtain an affine approximation of the measured acceleration about $a = 0$ and $x = (l/2)$. (10pts)

(b) Obtain an affine approximation of the measured acceleration about $a = 10$ and $x = l$. (10pts)

4. A control system is described in state-space representation, such that

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 5 & -2 \\ 8 & -3 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 2 \\ 4 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} -5 & 3 \end{bmatrix} \boldsymbol{x}(t) + u(t),$$

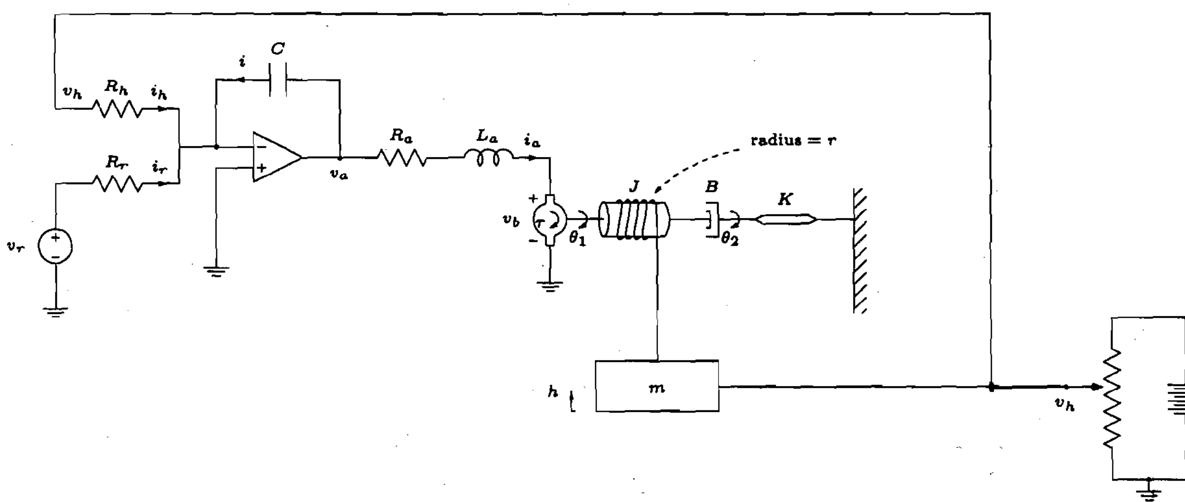
where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively.

(a) Solve $y(t)$ for $t \geq 0$, when $u(t) = 0$ for $t \geq 0$ and $\boldsymbol{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. (15pts)

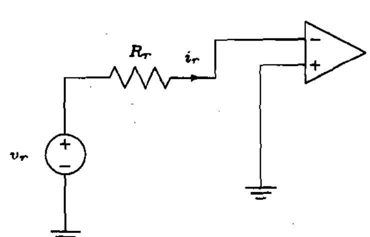
(b) Determine the transfer function or the transfer matrix of the system. (10pts)

EE 231 | **Exam#1 Solutions** | **Mar. 06, 2008**

1. In the following system, a motor is used to control the height of a massive structure. The height h is detected as a voltage signal v_h by the use of a variable resistor, such that $v_h = k_h h$. Obtain the detailed block diagram of the system, where v_r is the input and h is the output, and show the variables v_r, i_r, v_h, i_h, i, v_a, i_a, v_b, τ, θ_1, θ_2, and h on the block diagram.



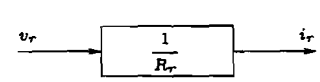
Solution: To determine the block diagram of the system, we first separate it into simpler components.

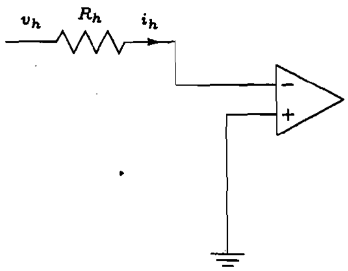


Since the input variable is v_r, we write i_r in terms v_r, such that

$$I_r(s) = \frac{1}{R_r} V_r(s),$$

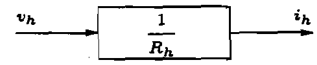
since the operational amplifier is assumed to be ideal.

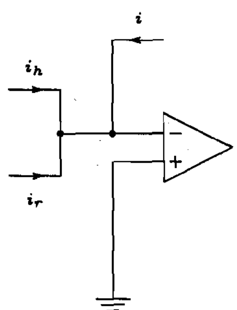




Similarly, we have

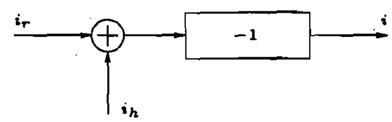
$$I_h(s) = \frac{1}{R_h} V_h(s).$$

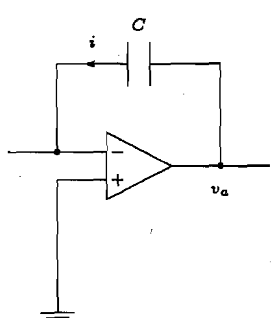




For an ideal operational amplifier,

$$i(t) = -\big(i_r(t) + i_h(t)\big).$$



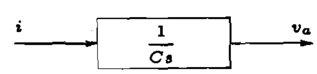


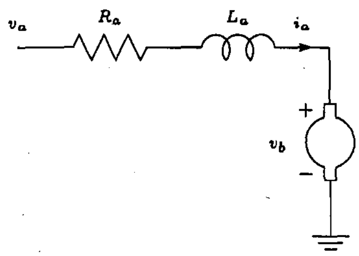
Again for an ideal operational amplifier,

$$v_a(t) = \frac{1}{C} \int^t i(\tau)\, \mathrm{d}\tau,$$

or

$$V_a(s) = \frac{1}{Cs} I(s).$$



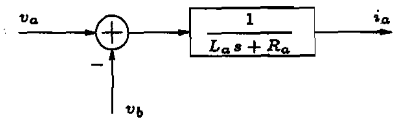


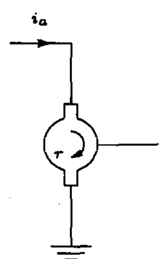
The armature current of the motor can be obtained from the Kirchhoff's Voltage Law, where

$$L_a \frac{\mathrm{d}i_a(t)}{\mathrm{d}t} + R_a i_a(t) + v_b(t) = v_a(t),$$

or

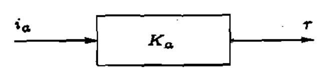
$$I_a(s) = \frac{1}{L_a s + R_a}\big(V_a(s) - V_b(s)\big).$$

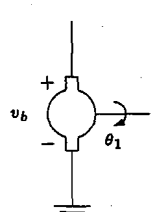




From the armature-controlled motor,

$$\tau(t) = K_a i_a(t).$$



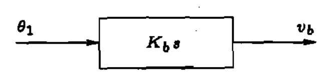


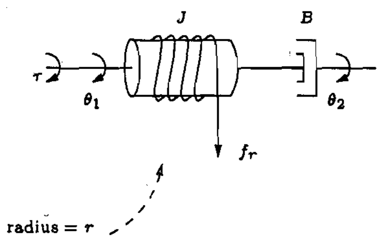
The back-emf voltage of the motor

$$v_b(t) = K_b \dot{\theta}_1(t),$$

or

$$V_b(s) = (K_b s)\Theta_1(s).$$



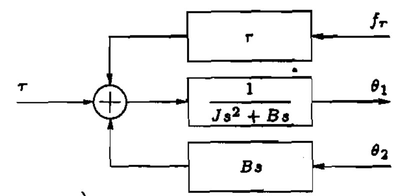


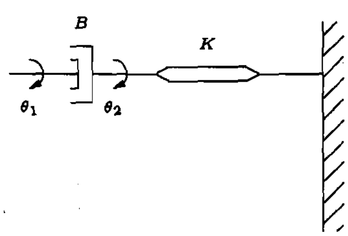
The torque equation for θ_1 is

$$J\ddot{\theta}_1(t) = \tau(t) + r f_r(t) - B\big(\dot{\theta}_1(t) - \dot{\theta}_2(t)\big),$$

where f_r is the internal tension of the rope. So,

$$\Theta_1(s) = \frac{1}{Js^2 + Bs}\Big(T(s) + rF_r(s) + Bs\Theta_2(s)\Big).$$



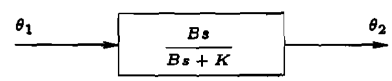


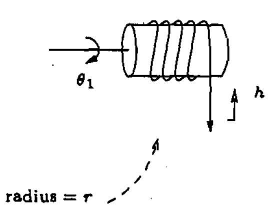
The torque equation for θ_2 is

$$0 = -B\big(\dot{\theta}_2(t) - \dot{\theta}_1(t)\big) - K\theta_2(t).$$

So,

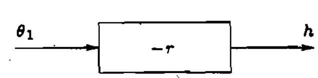
$$\Theta_2(s) = \frac{Bs}{Bs + K}\Theta_1(s).$$

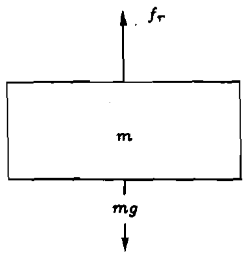




The disc with the inertia J changes the rotational motion to translational motion, where

$$h(t) = -r\theta_1(t).$$



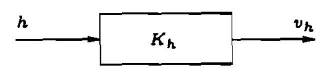


The force acting on the mass is only due to the gravity.

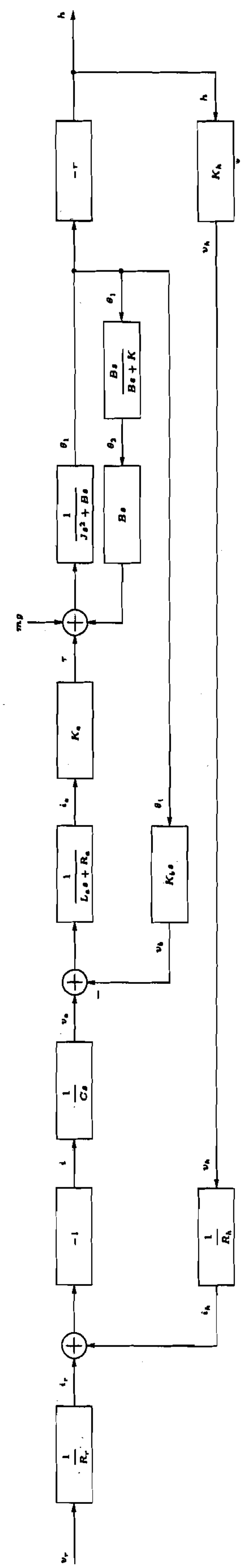
$$f_r(t) = mg\mathbf{1}(t).$$

And, finally the given relationship

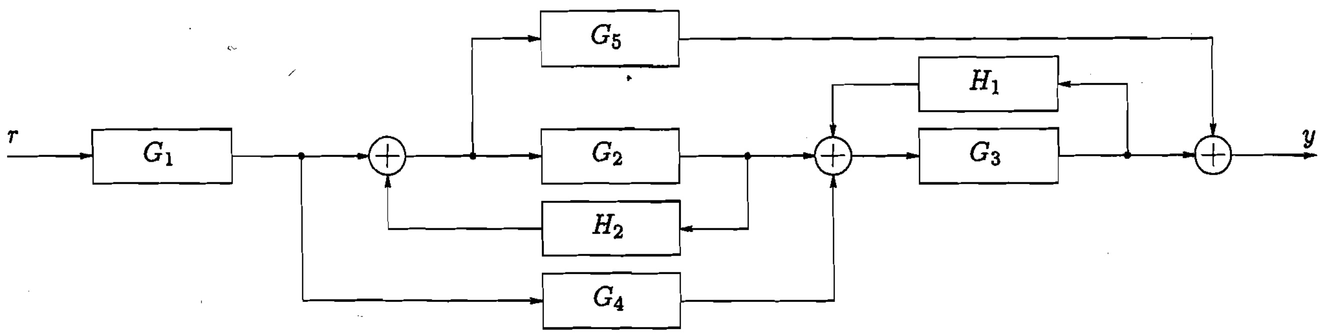
$$v_h(t) = K_h h(t).$$



When we connect all the individual blocks together, we get the following block diagram.

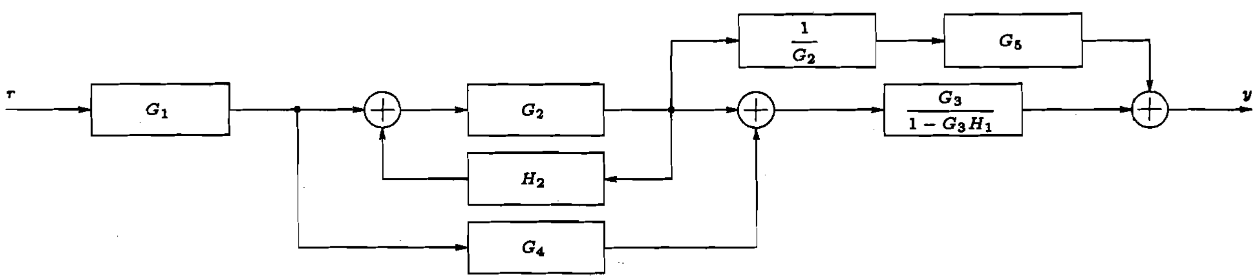


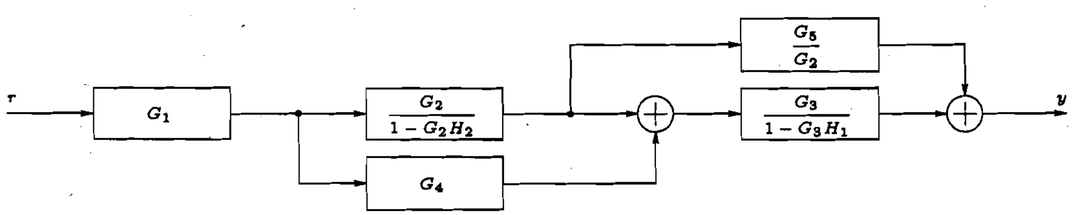
2. Consider the following block diagram of a control system.

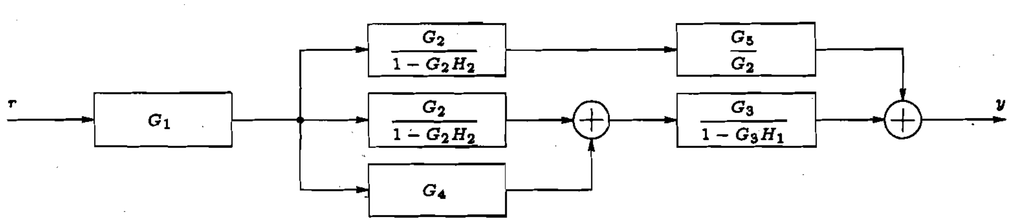


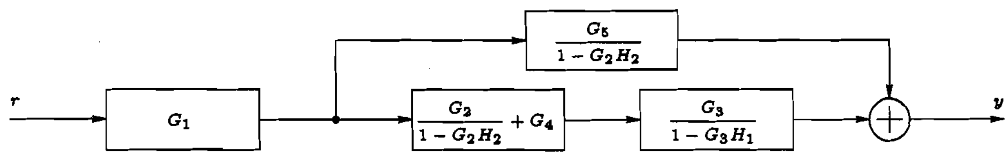
Determine the transfer function of the system. Show all your work.

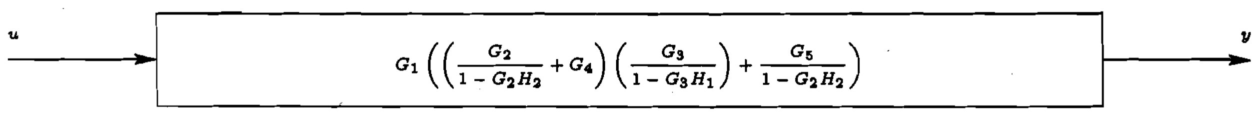
Solution: If we choose to use the block-diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.



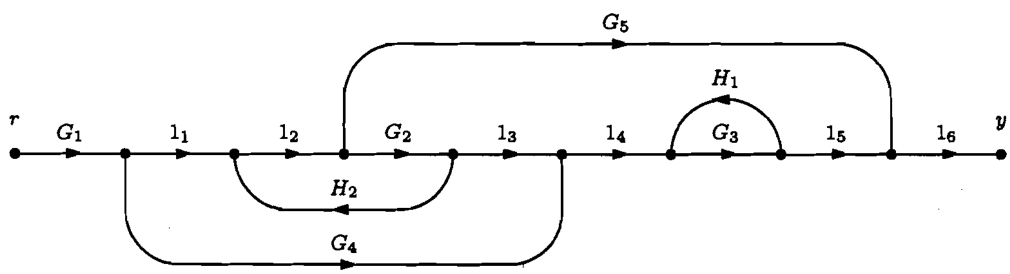








If we choose to use Mason's formula, we need to draw the signal flow graph of the block diagram.



In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The forward path gains are

$$F_1 = G_1 1_1 1_2 G_2 1_3 1_4 G_3 1_5 1_6 = G_1G_2G_3,$$

$$F_2 = G_1 G_4 1_4 G_3 1_5 1_6 = G_1G_4G_3,$$

$$F_3 = G_2 1_1 1_2 G_5 1_6 = G_1G_5.$$

The loop gains are

$$L_1 = 1_2 G_2 H_2 = G_2H_2,$$

$$L_2 = G_3H_1.$$

From the forward path and the loop gains, we determine the touching loops and the forward paths.

Touching Loops

	L_1	L_2
L_1	✔	✘
L_2		✔

Loops on Forward Paths

	L_1	L_2
F_1	✔	✔
F_2	✘	✔
F_3	✔	✘

Therefore,

$$\begin{aligned}\Delta &= 1 - \big(L_1 + L_2\big) + \big(L_1L_2\big)\\ &= 1 - \big((G_2H_2) + (G_3H_1)\big) + \big((G_2H_2)(G_3H_1)\big)\\ &= 1 - G_2H_2 - G_3H_1 + G_2H_2G_3H_1,\end{aligned}$$

and

$$\begin{aligned}\Delta_1 &= \Delta\big|_{L_1=L_2=0} = 1,\\ \Delta_2 &= \Delta\big|_{L_2=0} = 1 - L_1 = 1 - G_2H_2,\\ \Delta_3 &= \Delta\big|_{L_1=0} = 1 - L_2 = 1 - G_3H_1.\end{aligned}$$

So,

$$\frac{Y(s)}{U(s)} = \frac{1}{\Delta}\sum_{i=1}^{3} F_i\Delta_i = \frac{(G_1G_2G_3)(1) + (G_1G_4G_3)(1 - G_2H_2) + (G_1G_5)(1 - G_3H_1)}{1 - G_2H_2 - G_3H_1 + G_2H_2G_3H_1},$$

or

$$\frac{Y(s)}{U(s)} = \frac{G_1G_2G_3 + G_1G_4G_3(1 - G_2H_2) + G_1G_5(1 - G_3H_1)}{1 - G_2H_2 - G_3H_1 + G_2H_2G_3H_1}.$$

3. The acceleration measured by a sensing device that is placed along a flexible beam is given by

$$a_m(t) = \begin{cases} -a^2(t)\sin\big((\pi/(2l))x(t)\big), & \text{if } a(t) < 0;\\ a^2(t)\sin\big((\pi/(2l))x(t)\big), & \text{if } a(t) \geq 0;\end{cases}$$

where a and x are the actual acceleration and the displacement along the flexible beam of length l, respectively.

(a) Obtain an affine approximation of the measured acceleration about $a = 0$ and $x = (l/2)$.

Solution: The affine approximation is obtained from the taylor series expansion of $a_m(t)$ about $a = 0$ and $x = (l/2)$.

$$\begin{aligned} a_m &= \Big[\, a_m \,\Big]_{\substack{a=0\\ x=(l/2)}} + \left[\frac{\partial}{\partial a}\big(a_m\big)\right]_{\substack{a=0\\ x=(l/2)}} (a - 0) + \left[\frac{\partial}{\partial x}\big(a_m\big)\right]_{\substack{a=0\\ x=(l/2)}} (x - l/2) + \mathcal{O}(a^2, x^2)\\ &= \begin{cases} \big[\, -a^2\sin\big((\pi/(2l))x\big)\,\big]_{\substack{a=0\\ x=(l/2)}} + \big[-2a\sin\big((\pi/(2l))x\big)\big]_{\substack{a=0\\ x=(l/2)}} (a-0) \\ \qquad + \big[-a^2\big(\pi/(2l)\big)\cos\big((\pi/(2l))x\big)\big]_{\substack{a=0\\ x=(l/2)}} (x - l/2) + \mathcal{O}(a^2, x^2), & \text{if } a < 0;\\ \big[\, a^2\sin\big((\pi/(2l))x\big)\,\big]_{\substack{a=0\\ x=(l/2)}} + \big[2a\sin\big((\pi/(2l))x\big)\big]_{\substack{a=0\\ x=(l/2)}} (a-0) \\ \qquad + \big[a^2\big(\pi/(2l)\big)\cos\big((\pi/(2l))x\big)\big]_{\substack{a=0\\ x=(l/2)}} (x - l/2) + \mathcal{O}(a^2, x^2), & \text{if } a \geq 0;\end{cases}\\ &= \big[\,0\,\big] + \big[\,0\,\big](a - 0) + \big[\,0\,\big](x - l/2) + \mathcal{O}(a^2, x^2)\\ &\approx 0.\end{aligned}$$

As a result, the affine approximation of the measured acceleration about $a = 0$ and $x = (l/2)$ is

$$a_m(t) = 0.$$

(b) Obtain an affine approximation of the measured acceleration about $a = 10$ and $x = l$.

Solution: The affine approximation is obtained from the taylor series expansion of $a_m(t)$ about $a = 10$ and $x = l$.

$$a_m = \Big[\, a_m \,\Big]_{\substack{a=10\\x=l}} + \left[\frac{\partial}{\partial a}\Big(a_m\Big)\right]_{\substack{a=10\\x=l}} (a-10) + \left[\frac{\partial}{\partial x}\Big(a_m\Big)\right]_{\substack{a=10\\x=l}} (x-l) + \mathcal{O}(a^2, x^2)$$

$$= \begin{cases} \big[\, -a^2 \sin\big((\pi/(2l))x\big) \,\big]_{\substack{a=10\\x=l}} + \big[-2a \sin\big((\pi/(2l))x\big)\big]_{\substack{a=10\\x=l}} (a-10) \\ \qquad + \big[-a^2 (\pi/(2l)) \cos\big((\pi/(2l))x\big)\big]_{\substack{a=10\\x=l}} (x-l) + \mathcal{O}(a^2,x^2), & \text{if } a < 0; \\ \big[\, a^2 \sin\big((\pi/(2l))x\big) \,\big]_{\substack{a=10\\x=l}} + \big[2a \sin\big((\pi/(2l))x\big)\big]_{\substack{a=10\\x=l}} (a-10) \\ \qquad + \big[a^2 (\pi/(2l)) \cos\big((\pi/(2l))x\big)\big]_{\substack{a=10\\x=l}} (x-l) + \mathcal{O}(a^2,x^2), & \text{if } a \geq 0; \end{cases}$$

$$= \big[\, 100 \sin(\pi/2) \,\big] + \big[\, 20 \sin(\pi/2) \,\big](a-10) + \big[\, 100(\pi/(2l)) \cos(\pi/2) \,\big](x-l) + \mathcal{O}(a^2,x^2)$$

$$\approx 100 + 20(a-10).$$

As a result, the affine approximation of the measured acceleration about $a = 10$ and $x = l$ is

$$a_m(t) = 20a(t) - 100.$$

4. A control system is described in state-space representation, such that

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 5 & -2 \\ 8 & -3 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 2 \\ 4 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} -5 & 3 \end{bmatrix} \boldsymbol{x}(t) + u(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively.

(a) Solve $y(t)$ for $t \geq 0$, when $u(t) = 0$ for $t \geq 0$ and $\boldsymbol{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$.

Solution: The general solution to the state-space representation of a system described by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t)$$

is obtained from

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)} B\boldsymbol{u}(\tau)\, \mathrm{d}\tau,$$

where

$$e^{At} = \mathcal{L}_s^{-1}\big[\,(sI - A)^{-1}\,\big](t).$$

Here, I is the appropriately dimensioned identity matrix. In our case,

$$A = \begin{bmatrix} 5 & -2 \\ 8 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \qquad C = \begin{bmatrix} -5 & 3 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 \end{bmatrix},$$

and $u(t) = 0$ for $t \geq 0$. As a result, the integral term in the solution of $\boldsymbol{x}$ is identically zero. So,

$$\begin{aligned}
y(t) &= Ce^{At}\boldsymbol{x}(0) \\
&= \begin{bmatrix} -5 & 3 \end{bmatrix} \mathcal{L}_s^{-1}\left[(sI - A)^{-1}\right](t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -5 & 3 \end{bmatrix} \mathcal{L}_s^{-1}\left[\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -2 \\ 8 & -3 \end{bmatrix}\right)^{-1}\right](t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -5 & 3 \end{bmatrix} \mathcal{L}_s^{-1}\left[\left(\begin{bmatrix} s-5 & 2 \\ -8 & s+3 \end{bmatrix}\right)^{-1}\right](t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -5 & 3 \end{bmatrix} \mathcal{L}_s^{-1}\left[\frac{1}{(s-5)(s+3)-(-8)(2)} \begin{bmatrix} s+3 & -2 \\ 8 & s-5 \end{bmatrix}\right](t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -5 & 3 \end{bmatrix} \begin{bmatrix} \mathcal{L}_s^{-1}\left[\frac{s+3}{(s-1)^2}\right](t) & \mathcal{L}_s^{-1}\left[\frac{-2}{(s-1)^2}\right](t) \\ \mathcal{L}_s^{-1}\left[\frac{8}{(s-1)^2}\right](t) & \mathcal{L}_s^{-1}\left[\frac{s-5}{(s-1)^2}\right](t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -5 & 3 \end{bmatrix} \begin{bmatrix} \mathcal{L}_s^{-1}\left[\frac{-2}{(s-1)^2}\right](t) \\ \mathcal{L}_s^{-1}\left[\frac{-4}{(s-1)^2} + \frac{1}{s-1}\right](t) \end{bmatrix} \\
&= \begin{bmatrix} -5 & 3 \end{bmatrix} \begin{bmatrix} -2te^t \\ e^t - 4te^t \end{bmatrix}.
\end{aligned}$$

Or,

$$y(t) = -2te^t + 3e^t \text{ for } t \geq 0.$$

(b) Determine the transfer function or the transfer matrix of the system.

Solution: The transfer matrix of a control system described in the state-state representation

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{u}(t), \\
\boldsymbol{y}(t) &= C\boldsymbol{x}(t) + D\boldsymbol{u}(t),
\end{aligned}$$

is

$$F(s) = C(sI - A)^{-1}B + D,$$

where

$$A = \begin{bmatrix} 5 & -2 \\ 8 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ 4 \end{bmatrix},$$

$$C = \begin{bmatrix} -5 & 3 \end{bmatrix}, \qquad D = 1,$$

and I is the appropriately dimensioned identity matrix. So,

$$
\begin{aligned}
F(s) &= \begin{bmatrix} -5 & 3 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -2 \\ 8 & -3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 1 \\
&= \begin{bmatrix} -5 & 3 \end{bmatrix} \left(\begin{bmatrix} s-5 & 2 \\ -8 & s+3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 1 \\
&= \frac{1}{(s-5)(s+3)-(-8)(2)} \begin{bmatrix} -5 & 3 \end{bmatrix} \begin{bmatrix} s+3 & -2 \\ 8 & s-5 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 1 \\
&= \frac{1}{(s-1)^2}(2s-2) + 1 \\
&= \frac{2}{s-1} + 1.
\end{aligned}
$$

Therefore, the transfer matrix is

$$F(s) = \frac{s+1}{s-1}.$$