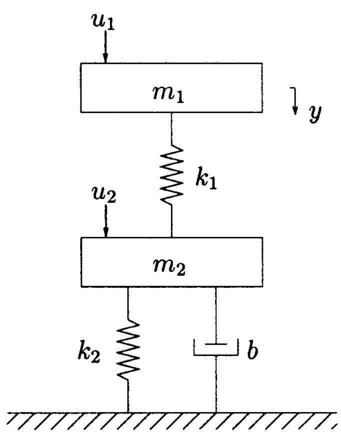
EE 231 **Exam#1** Oct. 17, 2002

75 minutes

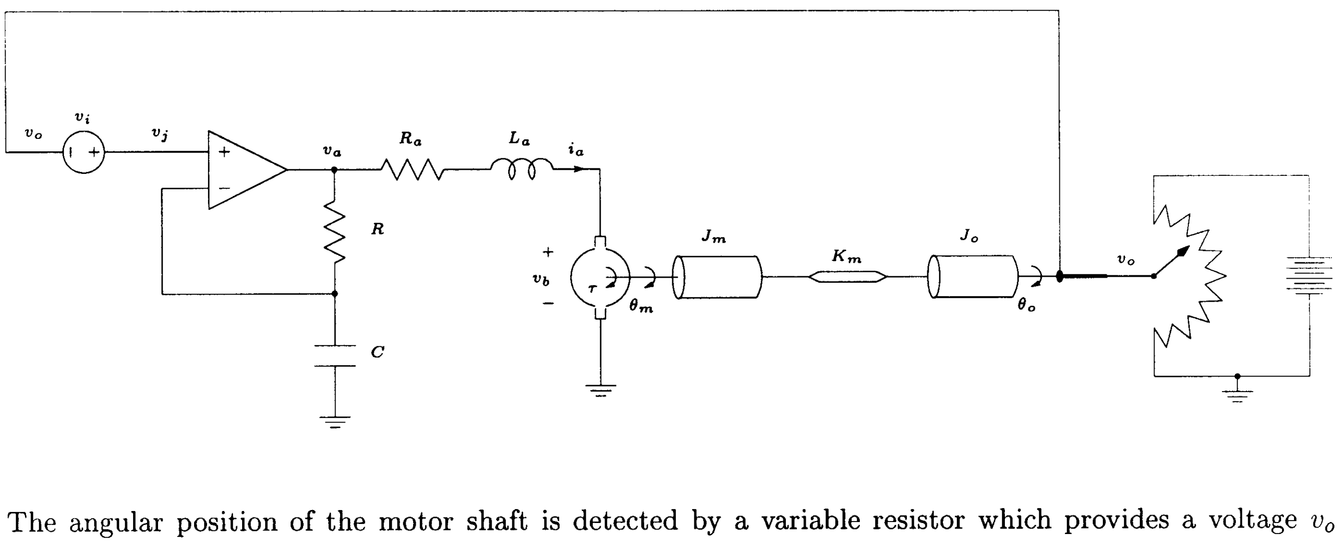
1. Consider the mechanical system shown below.



(a) Obtain the tranfer matrix of the system assuming that the external forces u_1 and u_2 are the inputs, and the displacement y is the output. (15pts)

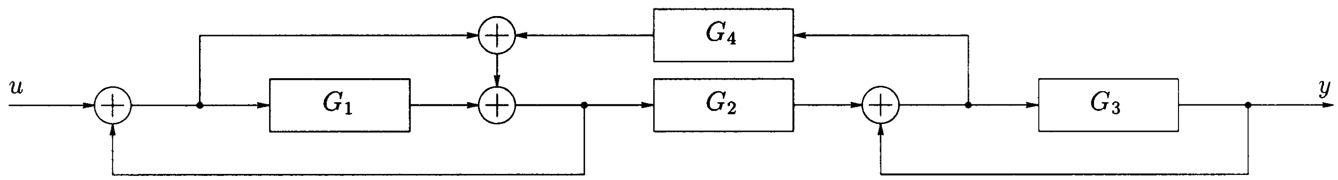
(b) Obtain *either* the force-voltage *or* the force-current analog of the system. (10pts)

2. The angular position of the shaft of a motor is controlled by the system shown below.



The angular position of the motor shaft is detected by a variable resistor which provides a voltage v_o proportional to the angle, such that $v_o = -K_o\theta_o$. Draw the most detailed block diagram of the system, where v_i is the input, and θ_o is the output. Show all the variables v_i, v_o, v_j, v_a, v_b, i_a, τ, θ_m, and θ_o on the block diagram. (25pts)

3. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly. (25pts)



4. A control system is represented by

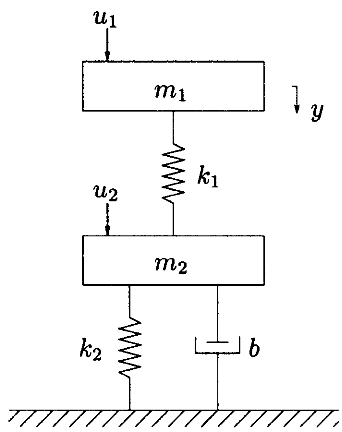
$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \end{bmatrix} u(t).$$

Determine $y(t)$ for $t \geq 0$; when $\mathbf{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, and $u(t) = 1$ for $t \geq 0$. (25pts)

EE 231 | **Exam#1 Solutions** | Oct. 17, 2002

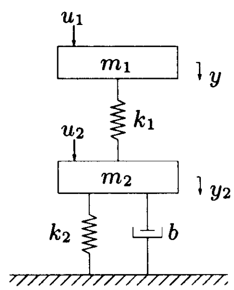
1. Consider the mechanical system shown below.



(a) Obtain the tranfer matrix of the system assuming that the external forces u_1 and u_2 are the inputs, and the displacement y is the output.

Solution:

First, we identify the linearly independent displacement locations in the mechanical system and mark them.



Then, we write the differential equations describing the motion from the mechanical system.

$$m_1\ddot{y} = u_1 - k_1(y - y_2)$$

$$m_2\ddot{y}_2 = u_2 - k_1(y_2 - y) - k_2y_2 - b\dot{y}_2.$$

Next, we obtain the transfer function by taking the Laplace transforms of the above equations under zero initial conditions. After some manipulations, we get

$$(m_1s^2 + k_1)Y(s) - k_1Y_2(s) = U_1(s),$$

and

$$-k_1Y(s) + (m_2s^2 + bs + k_1 + k_2)Y_2(s) = U_2(s),$$

where U_1, U_2, Y, and Y_2 are the Laplace transforms of u_1, u_2, y, and y_2, respectively. After multiplying the second equation by k_1 and substituting $k_1Y_2(s)$ from the first equation, we get

$$-k_1^2Y(s) + (m_2s^2 + bs + k_1 + k_2)\big((m_1s^2 + k_1)Y(s) - U_1(s)\big) = k_1U_2(s),$$

or

$$\big((m_2s^2 + bs + k_1 + k_2)(m_1s^2 + k_1) - k_1^2\big)Y(s) = (m_2s^2 + bs + k_1 + k_2)U_1(s) + k_1U_2(s).$$

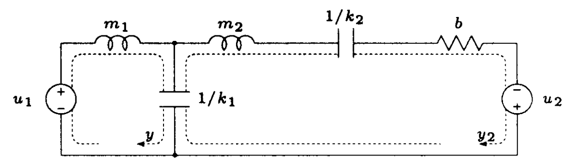
Therefore,

$$Y(s) = \frac{1}{(m_2s^2 + bs + k_1 + k_2)(m_1s^2 + k_1) - k_1^2}\begin{bmatrix} m_2s^2 + bs + k_1 + k_2 & k_1 \end{bmatrix}\begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}.$$

(b) Obtain *either* the force-voltage *or* the force-current analog of the system.

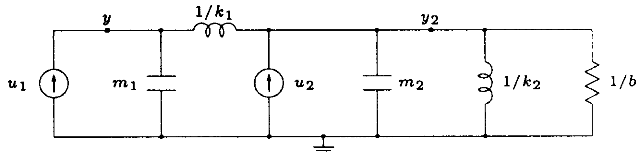
Solution: For the force-voltage analog of a mechanical system, there will be a loop charge associated with each displacement variable (or a loop current associated with each velocity variable), and an input force will be associated with a voltage source. The spring constant, the damping constant, and the mass will be associated with the reciprocal of capacitance, the resistance, and the inductance, respectively. The elements between two displacement variables of the mechanical system will be between the corresponding loop variables of the force-voltage analog. The elements that are connected to fixed frames and the elements that are always measured with respect to a fixed frame, such as the mass and the external force, will be on the non-common portions of the loops.

The next figure shows the force-voltage analog of the mechanical system, where the loops are identified with the displacements.

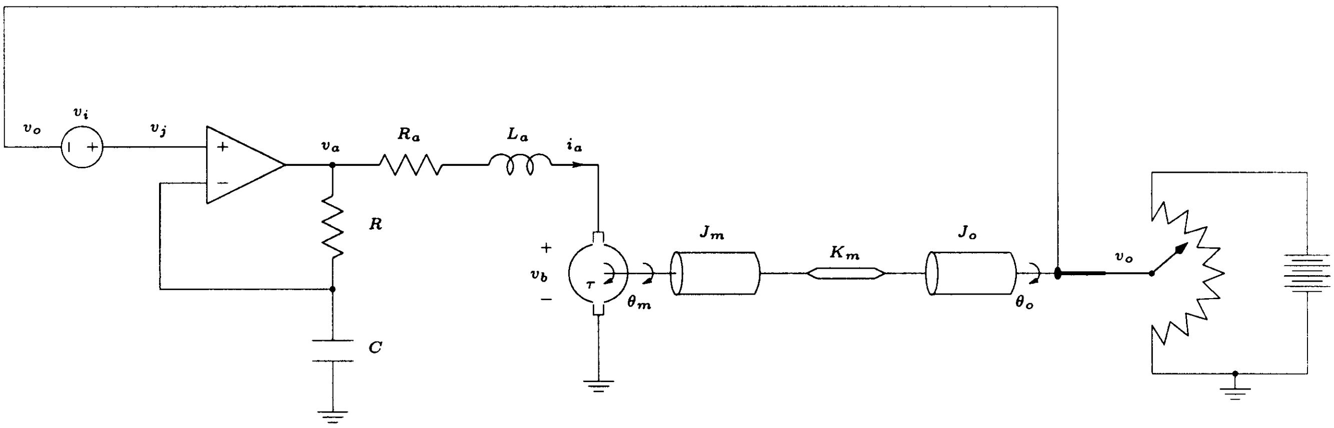


For the force-current analog of a mechanical system, there will be a node flux associated with each displacement variable (or a node voltage associated with each velocity variable), and an input force will be associated with a current source. The spring constant, the damping constant, and the mass will be associated with the reciprocal of inductance, the conductance, and the capacitance, respectively. The elements between two displacement variables of the mechanical system will be between the corresponding node variables of the force-voltage analog. The elements that are connected to fixed frames and the elements that are always measured with respect to a fixed frame, such as the mass and the external force, will be connected to the ground.

The next figure shows the force-current analog of the mechanical system, where the nodes are identified with the displacements.

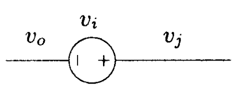


2. The angular position of the shaft of a motor is controlled by the system shown below.



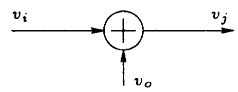
The angular position of the motor shaft is detected by a variable resistor which provides a voltage v_o proportional to the angle, such that $v_o = -K_o\theta_o$. Draw the most detailed block diagram of the system, where v_i is the input, and θ_o is the output. Show all the variables v_i, v_o, v_j, v_a, v_b, i_a, τ, θ_m, and θ_o on the block diagram.

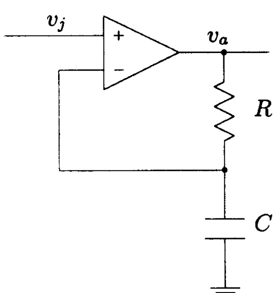
Solution: To determine the block diagram of the system, we first separate it into simpler components.



Because the input variable is v_i, we would write either v_j or v_o in terms of v_i, such that

$$v_j(t) = v_i(t) + v_o(t).$$



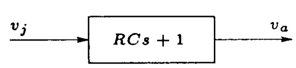


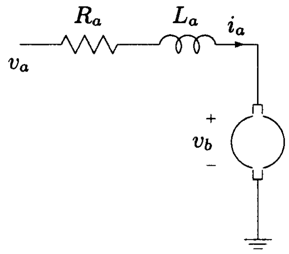
Since the operational amplifier is assumed to be ideal, we get

$$\frac{V_a(s)}{R + 1/(Cs)} = \frac{V_j(s)}{1/(Cs)},$$

or

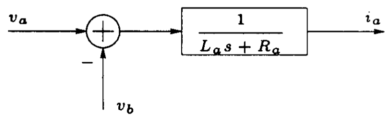
$$V_a(s) = (RCs + 1)V_j(s).$$

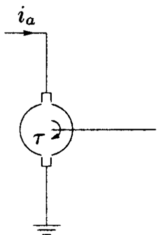




The armature current of the motor is

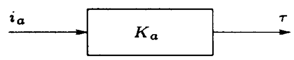
$$I_a(s) = \frac{1}{L_a s + R_a}\big(V_a(s) - V_b(s)\big).$$

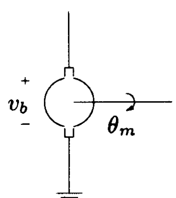




From the armature-controlled motor,

$$\tau(t) = K_a i_a(t).$$



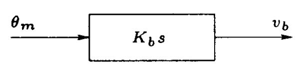


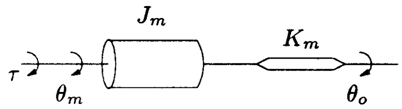
The back-emf voltage of the motor

$$v_b(t) = K_b \frac{\mathrm{d}\theta_m(t)}{\mathrm{d}t},$$

or

$$V_b(s) = (K_b s)\Theta_m(s).$$



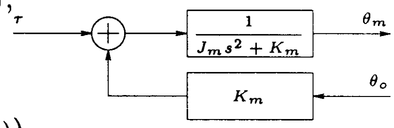


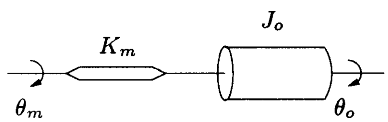
The torque equation for θ_m is

$$J_m \frac{\mathrm{d}^2\theta_m(t)}{\mathrm{d}t^2} = \tau(t) - K_m\big(\theta_m(t) - \theta_o(t)\big),$$

or

$$\Theta_m(s) = \frac{1}{J_m s^2 + K_m}\big(T(s) + K_m\Theta_o(s)\big).$$



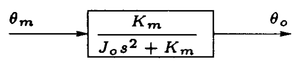


The torque equation for θ_o is

$$J_o \frac{\mathrm{d}^2\theta_o(t)}{\mathrm{d}t^2} = -K_m\big(\theta_o(t) - \theta_m(t)\big),$$

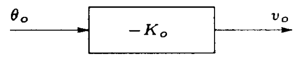
or

$$\Theta_o(s) = \frac{K_m}{J_o s^2 + K_m}\Theta_m(s).$$

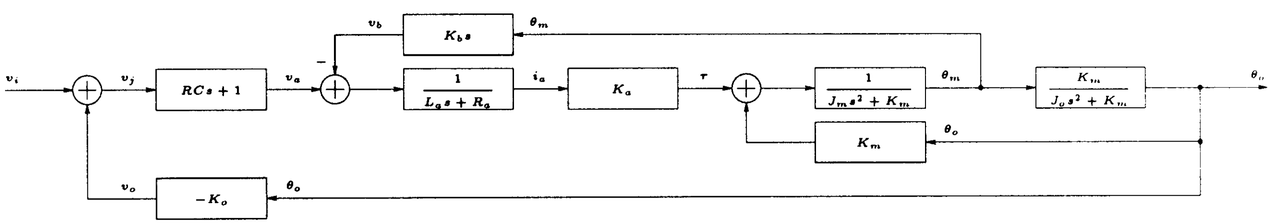


And, finally the given relationship

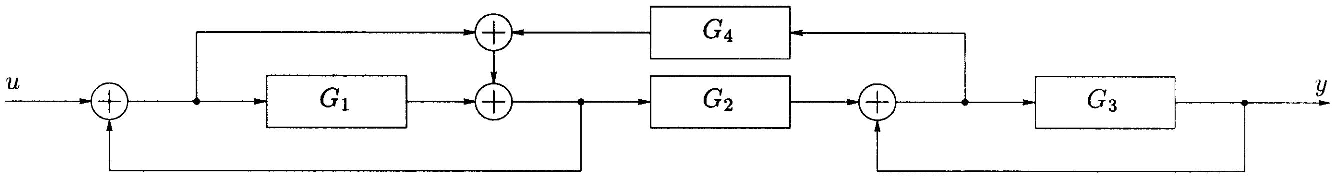
$$v_o(t) = -K_o\theta_o(t).$$



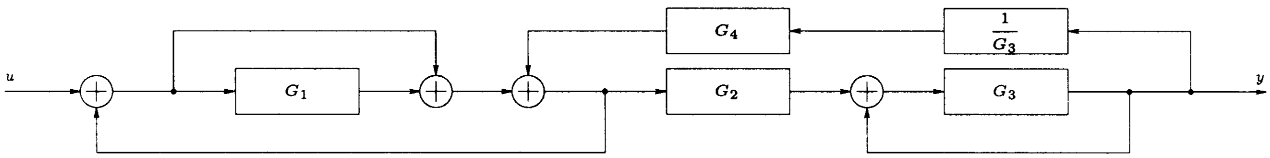
When we connect all the individual blocks together, we get the following block diagram.

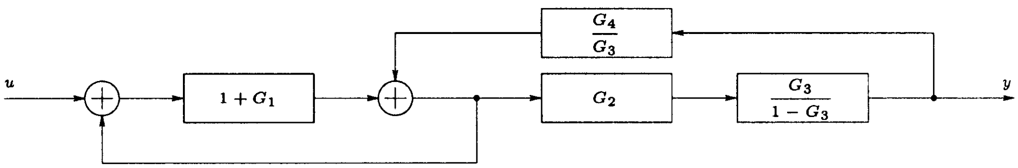


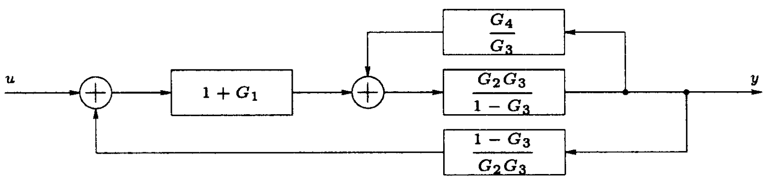
3. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly.

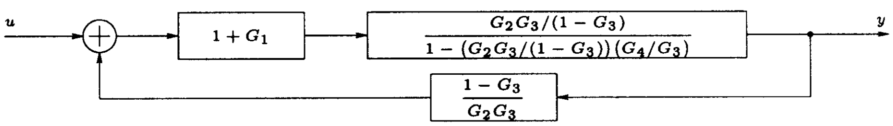


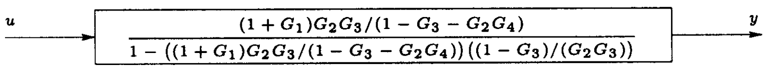
Solution: If we choose to use the block-diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.

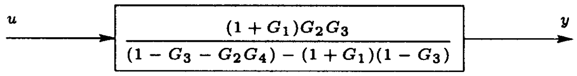




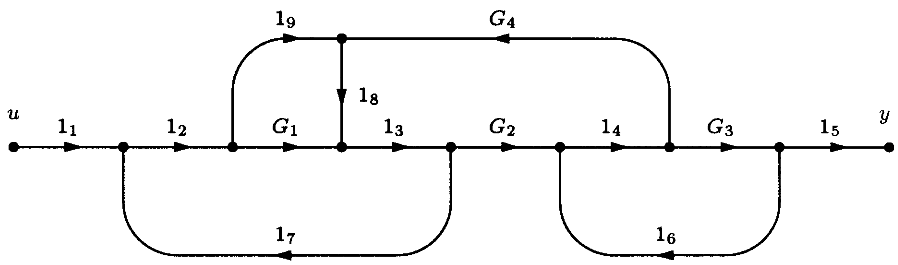








If we choose to use Mason's formula, we need to draw the signal flow graph of the block diagram.



In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The forward path gains are

$$F_1 = 1_1 1_2 G_1 1_3 G_2 1_4 G_3 1_5 = G_1 G_2 G_3,$$

and

$$F_2 = 1_1 1_2 1_9 1_8 1_3 G_2 1_4 G_3 1_5 = G_2 G_3.$$

The loop gains are

$$L_1 = 1_2 G_1 1_3 1_7 = G_1,$$
$$L_2 = 1_2 1_9 1_8 1_3 1_7 = 1,$$
$$L_3 = 1_3 G_2 1_4 G_4 1_8 = G_2 G_4,$$

and

$$L_4 = 1_4 G_3 1_6 = G_3.$$

From the forward path and the loop gains, we determine the touching loops and the forward paths.

Touching Loops

	L_1	L_2	L_3	L_4
L_1	✔	✔	✔	✘
L_2		✔	✔	✘
L_3			✔	✔
L_4				✔

Loops on Forward Paths

	L_1	L_2	L_3	L_4
F_1	✔	✔	✔	✔
F_2	✔	✔	✔	✔

Therefore,

$$\begin{aligned}\Delta &= 1 - \big(L_1 + L_2 + L_3 + L_4\big) + \big(L_1L_4 + L_2L_4\big)\\ &= 1 - \big((G_1) + (1) + (G_2G_4) + (G_3)\big) + \big((G_1)(G_3) + (1)(G_3)\big)\\ &= -G_1 - G_2G_4 - G_1G_3,\end{aligned}$$

and

$$\begin{aligned}\Delta_1 &= \Delta|_{L_1=L_2=L_3=L_4=0} = 1,\\ \Delta_2 &= \Delta|_{L_1=L_2=L_3=L_4=0} = 1.\end{aligned}$$

So,

$$\frac{Y(s)}{U(s)} = \frac{1}{\Delta}\sum_{i=1}^{2} F_i\Delta_i = \frac{(G_1G_2G_3)(1) + (G_2G_3)(1)}{-G_1 - G_2G_4 + G_1G_3},$$

or

$$\frac{Y(s)}{U(s)} = \frac{G_1G_2G_3 + G_2G_3}{-G_1 - G_2G_4 + G_1G_3}.$$

4. A control system is represented by

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}u(t),\\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 \end{bmatrix}u(t).\end{aligned}$$

Determine $y(t)$ for $t \geq 0$; when $\mathbf{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, and $u(t) = 1$ for $t \geq 0$.

Solution: The general solution to the state-space representation of a system described by

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t)\\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t)\end{aligned}$$

is obtained from

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)\,\mathrm{d}\tau,$$

where

$$e^{At} = \mathcal{L}_s^{-1}\left[(sI - A)^{-1}\right](t).$$

Here, I is the appropriately dimensioned identity matrix. In our case,

$$A = \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 \end{bmatrix},$$

$\mathbf{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, and $u(t) = 1$ for $t \geq 0$. As a result, the initial-condition term in the solution of x and the first term in the y equation are identically zero. So,

$$y(t) = C \int_0^t e^{A(t-\tau)} Bu(\tau)\, \mathrm{d}\tau + Du(t)$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \int_0^t \mathcal{L}_s^{-1} \left[(sI - A)^{-1} \right](t-\tau) \begin{bmatrix} 1 \\ -1 \end{bmatrix} (1)\, \mathrm{d}\tau + \begin{bmatrix} 1 \end{bmatrix} (1)$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \int_0^t \mathcal{L}_s^{-1} \left[\left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix} \right)^{-1} \right](t-\tau) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathrm{d}\tau + 1$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \int_0^t \mathcal{L}_s^{-1} \left[\left(\begin{bmatrix} s+1 & 0 \\ -4 & s+2 \end{bmatrix} \right)^{-1} \right](t-\tau) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathrm{d}\tau + 1$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \int_0^t \mathcal{L}_s^{-1} \left[\frac{1}{(s+1)(s+2) - (-4)(0)} \begin{bmatrix} s+2 & 0 \\ 4 & s+1 \end{bmatrix} \right](t-\tau) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathrm{d}\tau + 1$$

$$= \int_0^t \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{L}_s^{-1} \left[\frac{1}{s+1} \right](t-\tau) & 0 \\ \mathcal{L}_s^{-1} \left[\frac{4}{(s+1)(s+2)} \right](t-\tau) & \mathcal{L}_s^{-1} \left[\frac{1}{s+2} \right](t-\tau) \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathrm{d}\tau + 1$$

$$= \int_0^t \begin{bmatrix} \mathcal{L}_s^{-1} \left[\frac{1}{s+1} \right](t-\tau) & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathrm{d}\tau + 1$$

$$= \int_0^t \mathcal{L}_s^{-1} \left[\frac{1}{s+1} \right](t-\tau)\, \mathrm{d}\tau + 1$$

$$= \int_0^t e^{-(t-\tau)}\, \mathrm{d}\tau + 1$$

$$= \left(e^{-(t-\tau)} \right)_{\tau=0}^{\tau=t} + 1$$

$$= \left(1 - e^{-t} \right) + 1.$$

Or,

$$y(t) = 2 - e^{-t} \text{ for } t \geq 0.$$