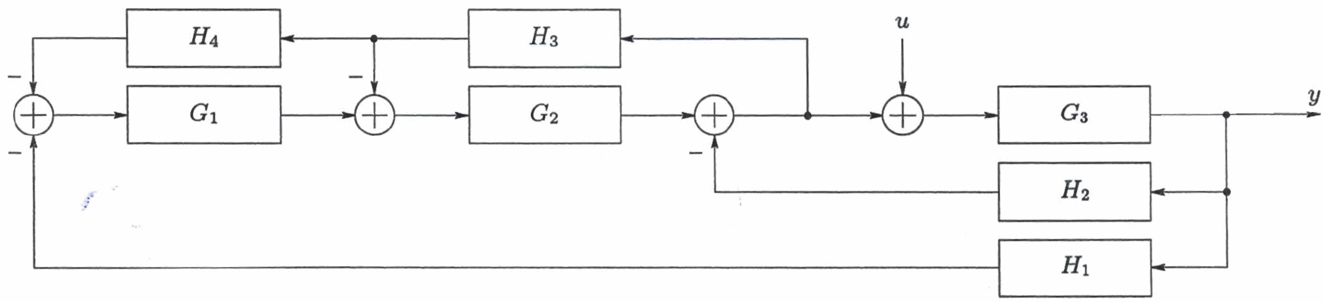
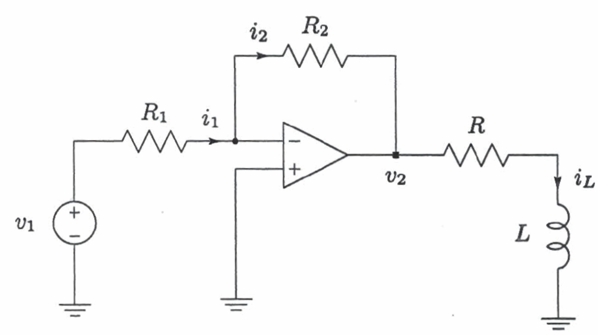
EE 231 — Exam#1 — Mar. 08, 2007

75 minutes

1. For the block diagram given below, determine the transfer function; where u is the input, and y is the output. Show your work clearly. (20pts)



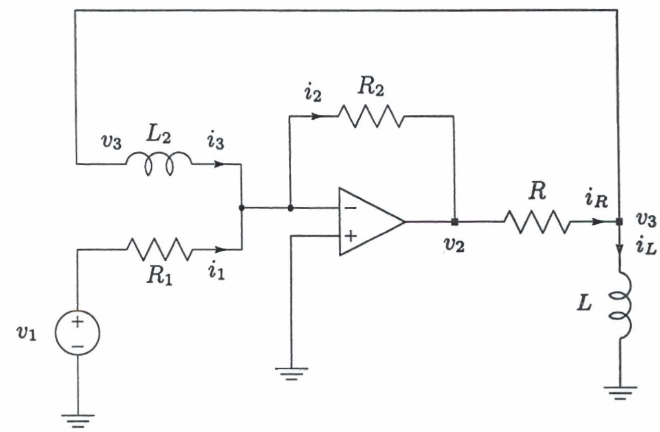
2. The field of a DC servomotor is excited by means of an amplifier as shown in the following figure, where the voltage v_1 and the current i_L are the input and the output variables, respectively. Here, $R_1 = 100\,\Omega$, $R_2 = 1\,\mathrm{k}\Omega$, $R = 50\,\Omega$, and $L = 2\,\mathrm{H}$.



(a) Calculate the time constant of this system. (10pts)

(b) Determine the 5% settling time for the unit-step input. (05pts)

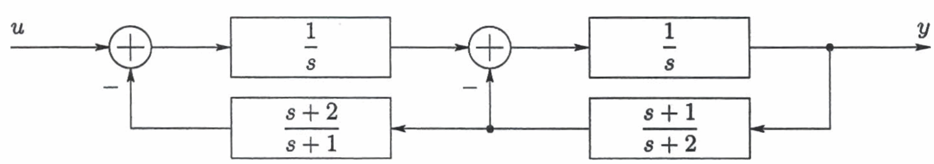
3. To improve the time response of the system, an inductor connection is fed-back to the amplifier as shown in the following figure.



(a) Draw the most detailed block diagram of the system, where v_1 is the input, and i_L is the output. Show all the variables v_1, i_1, i_2, v_2, i_R, i_L, v_3, and i_3 on the block diagram. (20pts)

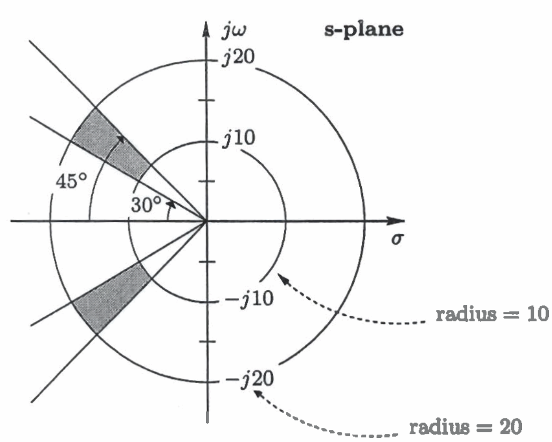
(b) Determine the inductance L_2, such that the time constant is reduced to its (1/4)th value. (10pts)

4. The block diagram of a control system is given below.



Obtain a state-space representation of the system without any block-diagram reduction. (20pts)

5. Consider a second-order system described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, such that its poles are located in the shaded region below.



Determine the smallest possible maximum percent-overshoot, the smallest possible peak time, and the smallest possible 2% settling-time of the system. (15pts)

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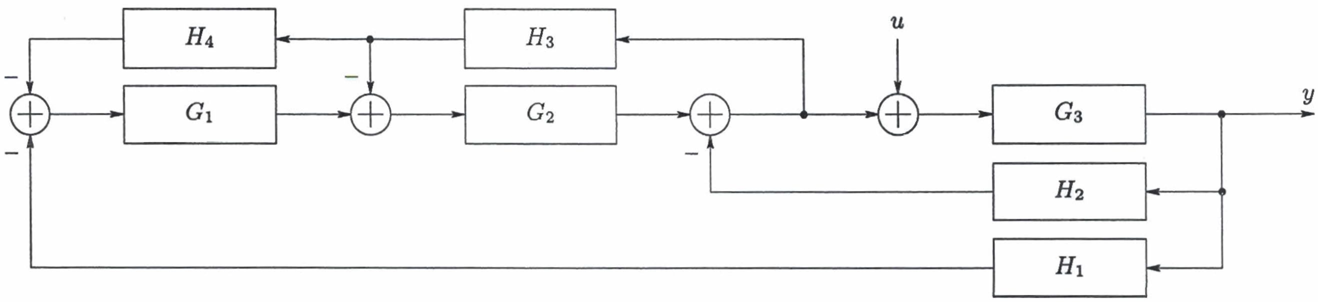
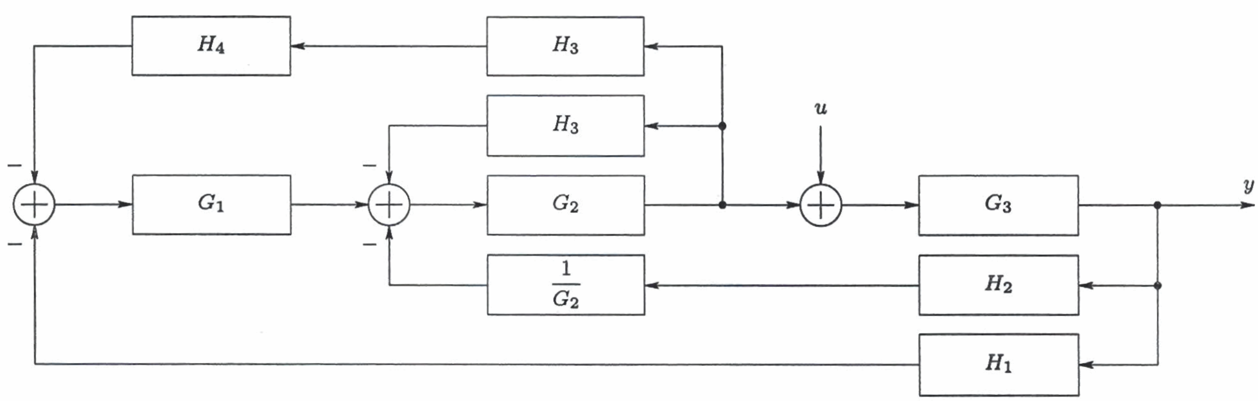
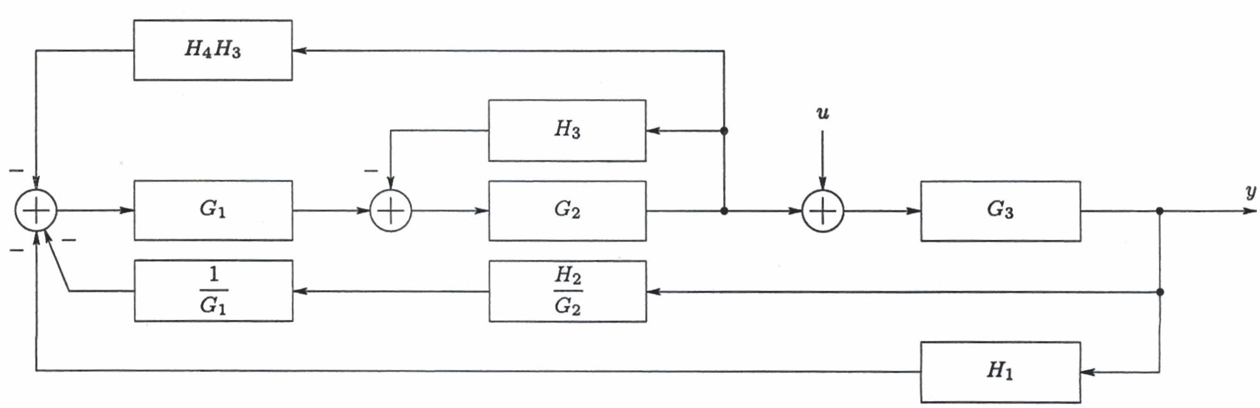
Solutions

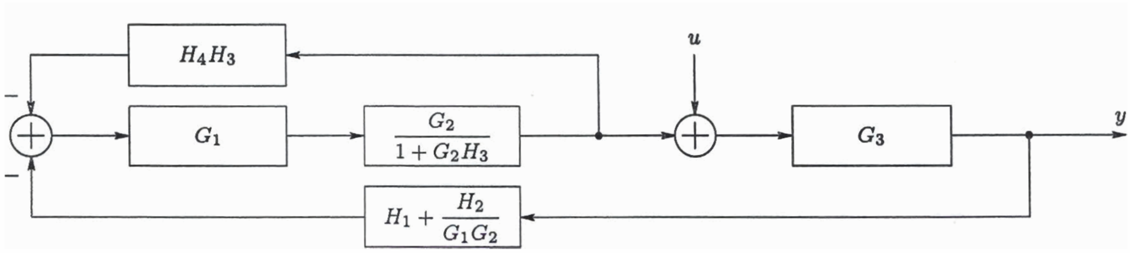
1. For the block diagram given below, determine the transfer function; where u is the input, and y is the output. Show your work clearly.

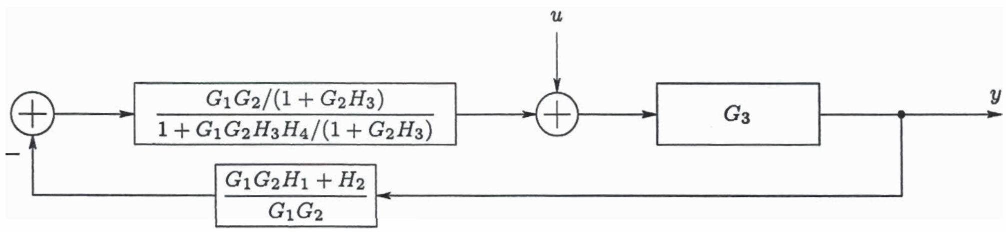


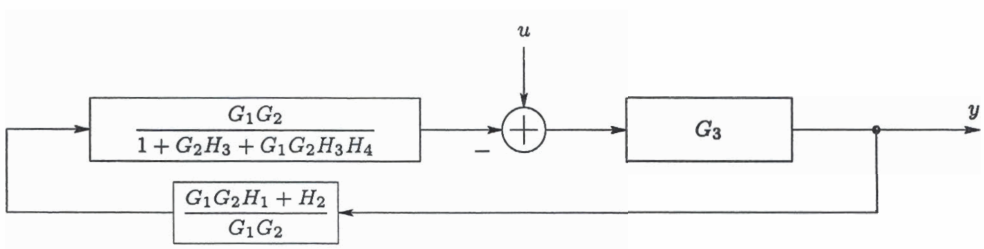
Solution: If we choose to use the block-diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.

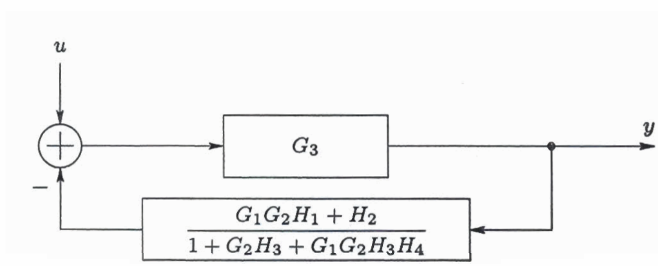


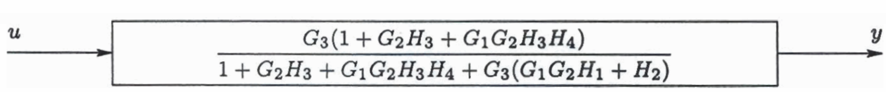




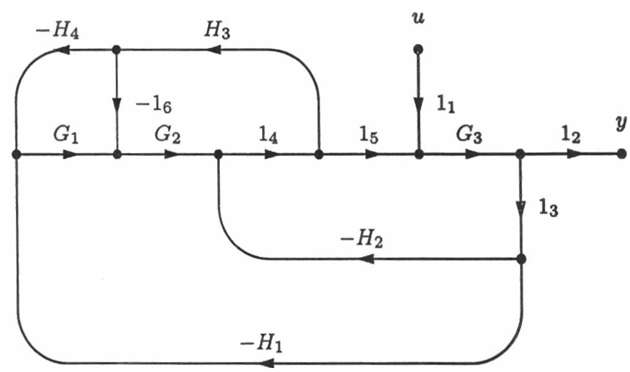








If we choose to use Mason's formula, we need to draw the signal flow graph of the block diagram.



In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The only forward path gain is

$$F_1 = 1_1 G_3 1_2 = G_3.$$

The loop gains are

$$\begin{aligned} L_1 &= 1_3(-H_1)G_1G_2 1_4 1_5 G_3 = -G_1G_2G_3H_1, \\ L_2 &= 1_3(-H_2)1_4 1_5 G_3 = -G_3H_2, \\ L_3 &= G_2 1_4 H_3(-1_6) = -G_2H_3, \\ L_4 &= G_1G_2 1_4 H_3(-H_4) = -G_1G_2H_3H_4. \end{aligned}$$

From the forward path and the loop gains, we determine the touching loops and the forward path.

Touching Loops

	L_1	L_2	L_3	L_4
L_1	✔	✔	✔	✔
L_2		✔	✔	✔
L_3			✔	✔
L_4				✔

Loops on Forward Paths

	L_1	L_2	L_3	L_4
F_1	✔	✔	✘	✘

Therefore,

$$\begin{aligned} \Delta &= 1 - (L_1 + L_2 + L_3 + L_4) \\ &= 1 - ((-G_1G_2G_3H_1) + (-G_3H_2) + (-G_2H_3) + (-G_1G_2H_3H_4)) \\ &= 1 + G_1G_2G_3H_1 + G_3H_2 + G_2H_3 + G_1G_2H_3H_4, \end{aligned}$$

and

$$\Delta_1 = \Delta|_{L_1=L_2=0} = 1 - L_3 - L_4 = 1 + G_2H_3 + G_1G_2H_3H_4.$$

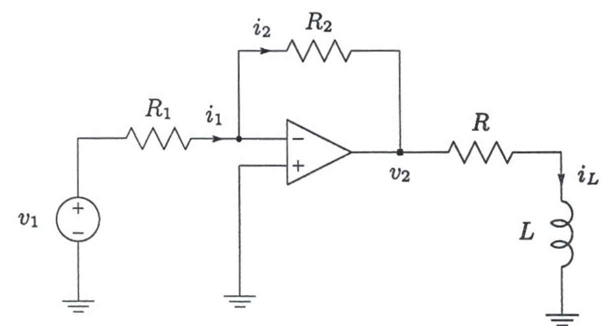
So,

$$\frac{Y(s)}{U(s)} = \frac{F_1\Delta_1}{\Delta} = \frac{(G_3)(1 + G_2H_3 + G_1G_2H_3H_4)}{1 + G_1G_2G_3H_1 + G_3H_2 + G_2H_3 + G_1G_2H_3H_4},$$

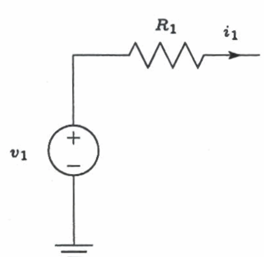
or

$$\frac{Y(s)}{U(s)} = \frac{G_3 + G_2G_3H_3 + G_1G_2G_3H_3H_4}{1 + G_1G_2G_3H_1 + G_3H_2 + G_2H_3 + G_1G_2H_3H_4}.$$

2. The field of a DC servomotor is excited by means of an amplifier as shown in the following figure, where the voltage v_1 and the current i_L are the input and the output variables, respectively. Here, $R_1 = 100\,\Omega$, $R_2 = 1\,\mathrm{k}\Omega$, $R = 50\,\Omega$, and $L = 2\,\mathrm{H}$.

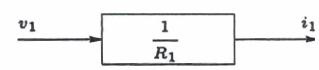


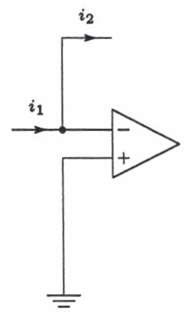
Solution: To determine the system equations or the block diagram of the system, we first separate it into simpler components.



Since the input variable is v_1, we write i_1 in terms v_1, such that

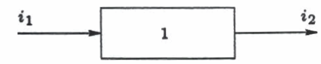
$$I_1(s) = \frac{1}{R_1}V_1(s).$$

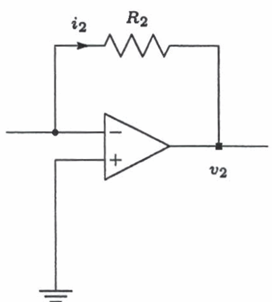




For an ideal operational amplifier,

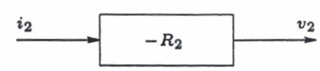
$$i_2(t) = i_1(t).$$

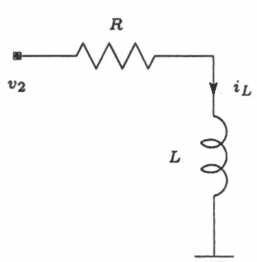




Again for an ideal operational amplifier,

$$V_2(s) = -R_2 I_2(s).$$



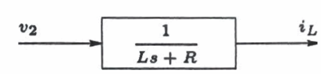


From Kirchhoff's Voltage Law, we have

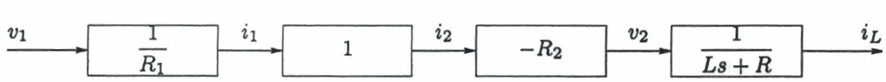
$$L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} + Ri_L(t) = v_2(t),$$

or

$$I_L(s) = \frac{1}{Ls + R} V_2(s).$$



When we connect all the individual blocks together, we get the following block diagram.



From the block diagram, we get the transfer function of the system.

$$\frac{I_L(s)}{V_1(s)} = \left(\frac{1}{Ls+R}\right)\left(-R_2\right)\left(1\right)\left(\frac{1}{R_1}\right) = -\frac{R_2/(R_1 L)}{s+R/L}.$$

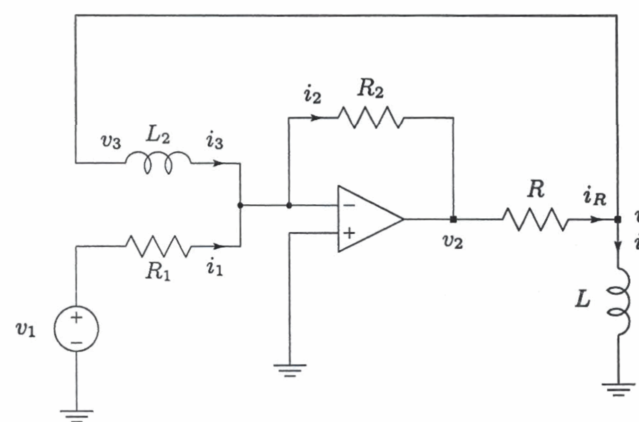
(a) Calculate the time constant of this system.

Solution: Since the system is a first-order system with its pole at $s = -(R/L)$, the time constant is $T = 1/(R/L) = (L/R)$. For $R = 50\,\Omega$ and $L = 2\,\text{H}$, we get $T = 0.04\,\text{s}$.

(b) Determine the 5% settling time for the unit-step input.

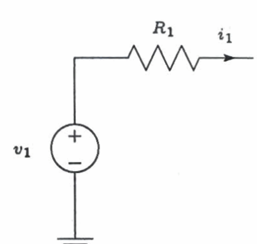
Solution: Since the system is first-order, the 5% settling time is three times the time constant, or $t_{5\%s} = 0.12\,\text{s}$.

3. To improve the time response of the system, an inductor connection is fed-back to the amplifier as shown in the following figure.



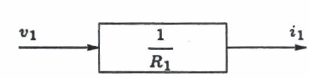
(a) Draw the most detailed block diagram of the system, where v_1 is the input, and i_L is the output. Show all the variables v_1, i_1, i_2, v_2, i_R, i_L, v_3, and i_3 on the block diagram.

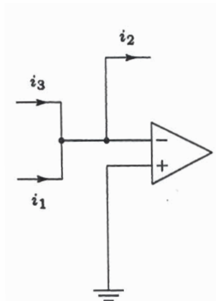
Solution: To determine the system equations for the feedback system, we again separate it into simpler components.



Since the input variable is v_1, we write i_1 in terms v_1, such that

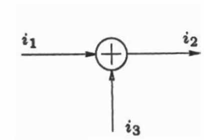
$$I_1(s) = \frac{1}{R_1}V_1(s).$$

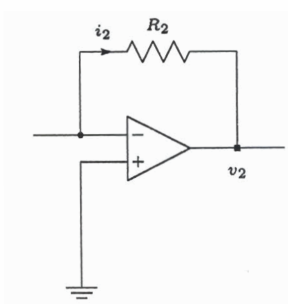




For an ideal operational amplifier,

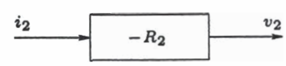
$$i_2(t) = i_1(t) + i_3(t).$$

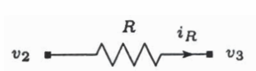




Again for an ideal operational amplifier,

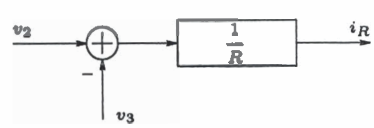
$$V_2(s) = -R_2 I_2(s).$$

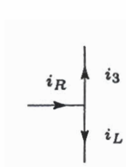




From Kirchhoff's Voltage Law, we have

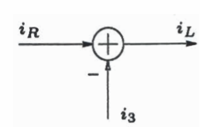
$$I_R(s) = \frac{1}{R}\big(V_2(s) - V_3(s)\big).$$

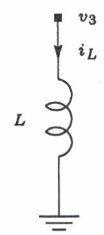




And,

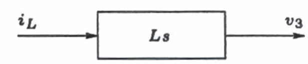
$$i_L(t) = i_R(t) - i_3(t).$$

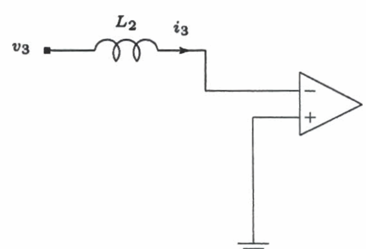




Also,

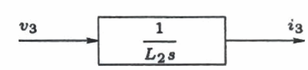
$$V_3(s) = (Ls)I_L(s).$$



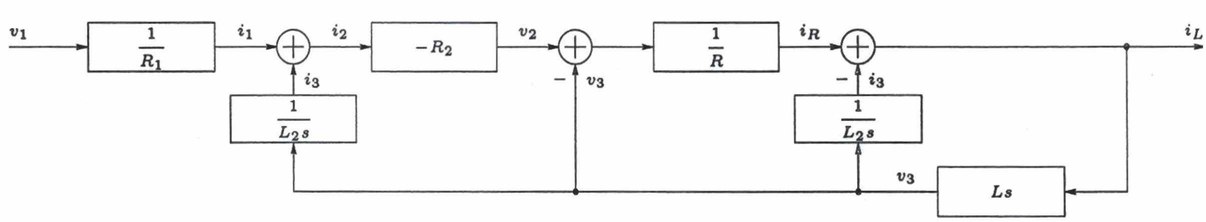


Finally, we have

$$I_3(s) = \frac{1}{L_2 s} V_3(s).$$

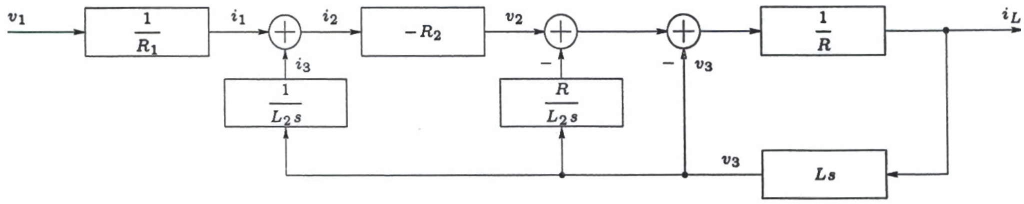


When we connect all the individual blocks together, we get the following block diagram.

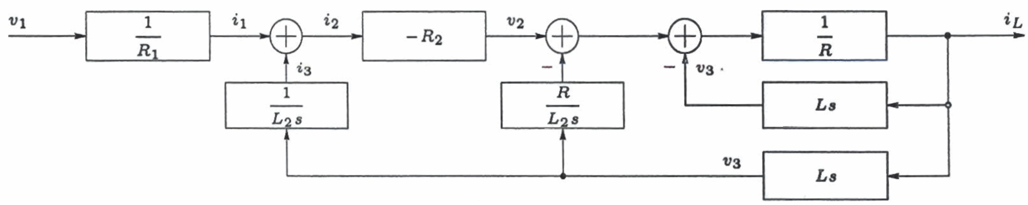


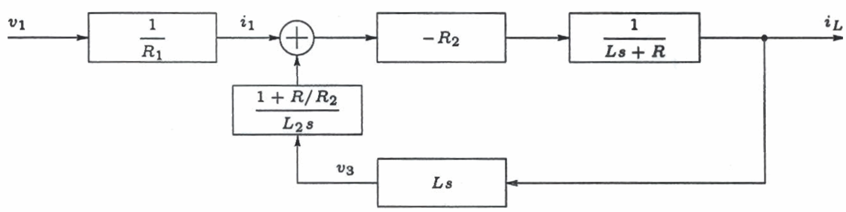
(b) Determine the inductance L_2, such that the time constant is reduced to its (1/4)th value.

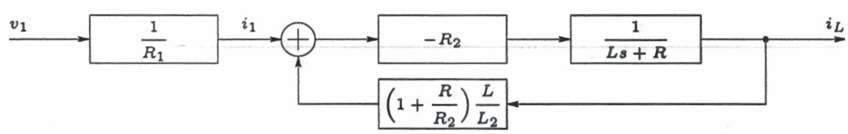
Solution: We may determine the new transfer function by reducing the block diagram. First, We may move the last loop behind the middle loop.



Next, we separate the last loop and simplify.







The new transfer function is

$$\frac{I_L(s)}{V_1(s)} = \left(\frac{1}{R_1}\right)\left(\frac{-R_2/(Ls+R)}{1+\big(R_2/(Ls+R)\big)\big((1+R/R_2)L/L_2\big)}\right)$$

$$= \frac{-R_2/R_1}{Ls+R+(R_2+R)L/L_2} = \frac{-R_2/(LR_1)}{s+\big(R/L+(R_2+R)/L_2\big)}.$$

Therefore, the new time constant

$$T_{\text{new}} = \frac{1}{R/L+(R_2+R)/L_2}.$$

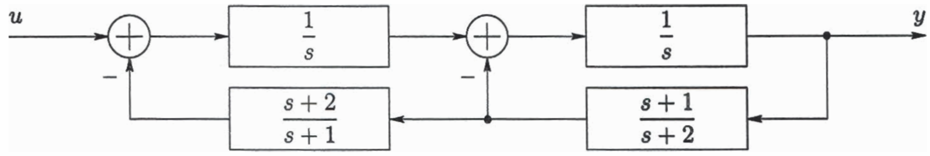
Setting the new time constant to (1/4)th of the original value, we get

$$\frac{1}{R/L+(R_2+R)/L_2} = \frac{1}{4}(0.04),$$

$$\frac{1}{50/2+(1000+50)/L_2} = 0.01,$$

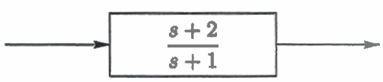
or $L_2 = 14\,\text{H}$.

4. The block diagram of a control system is given below.

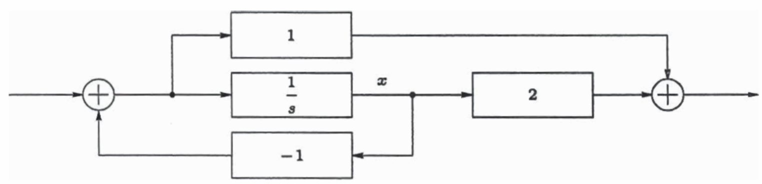


Obtain a state-space representation of the system without any block-diagram reduction.

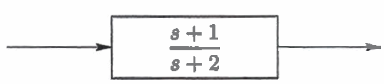
Solution: In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.



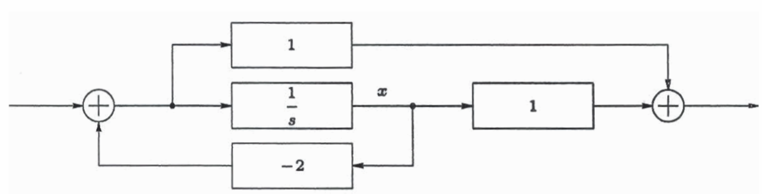
(a) The first feedback gain block.



(b) Controller realization form.

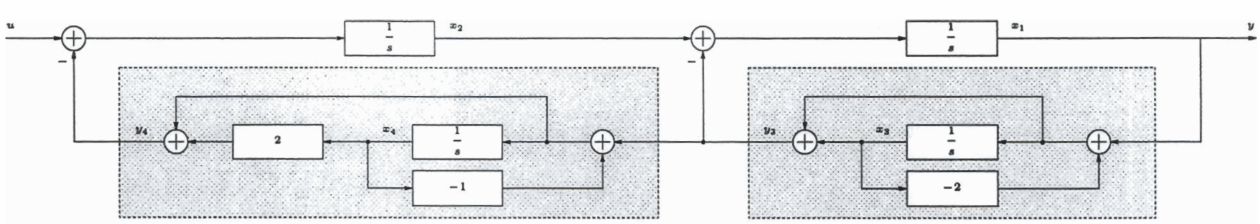


(a) The second feedback gain block.



(b) Controller realization form.

The connected and "expanded" block diagram is shown below.



After assigning the state variables as shown in the figure, we obtain

$$\dot{x}_1 = x_2 - y_3,$$
$$\dot{x}_2 = u - y_4,$$
$$\dot{x}_3 = -2x_3 + x_1,$$
$$\dot{x}_4 = -x_4 + y_3,$$

and

$$y = x_1;$$

where

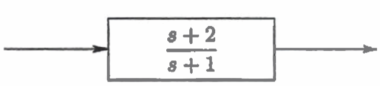
$$y_3 = x_3 + (-2x_3 + x_1) = x_1 - x_3,$$
$$y_4 = 2x_4 + (-x_4 + y_3) = x_1 - x_3 + x_4.$$

After substituting the y_3 and y_4 expressions into the differential equations and writing them in matrix form, we obtain the state-space representation

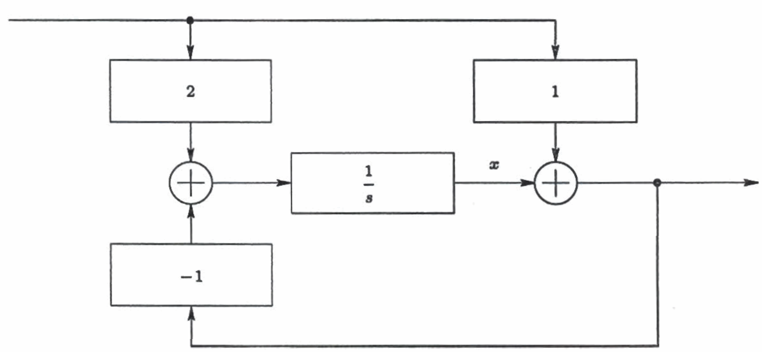
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & -1 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}.$$

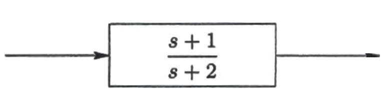
If we use the observer realization form for each of the blocks, then we obtain a different state-space representation.



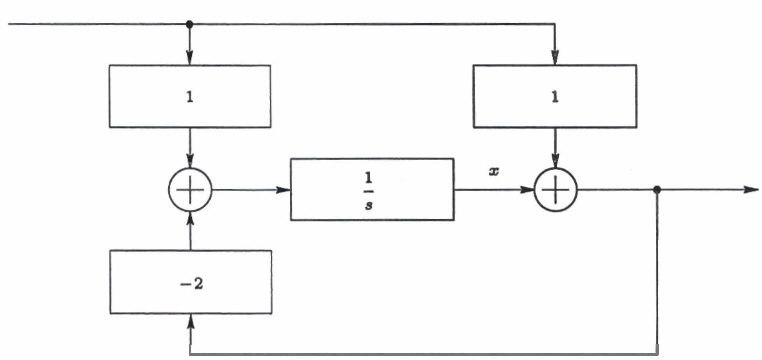
(a) The first feedback gain block.



(b) Observer realization form.

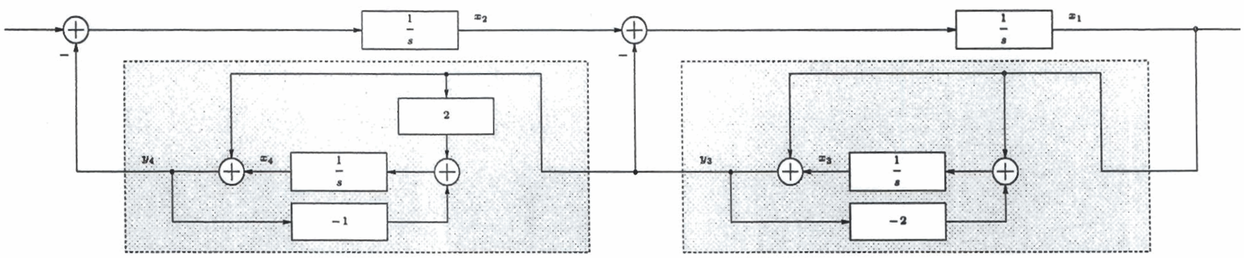


(a) The second feedback gain block.



(b) Observer realization form.

The connected and "expanded" block diagram for this case is shown below.



Similarly, we obtain

$$\dot{x}_1 = x_2 - y_3,$$
$$\dot{x}_2 = u - y_4,$$
$$\dot{x}_3 = -2y_3 + x_1,$$
$$\dot{x}_4 = -y_4 + 2y_3,$$

and

$$y = x_1;$$

where

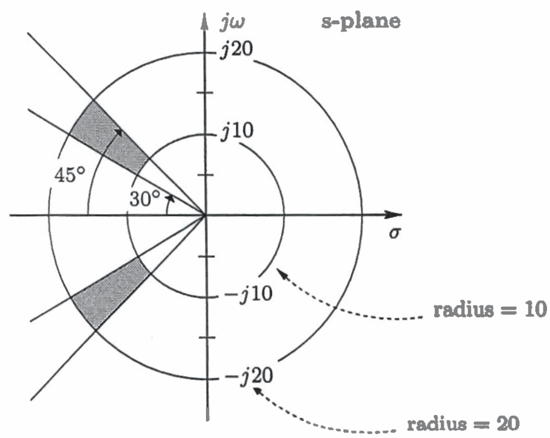
$$y_3 = x_3 + x_1,$$
$$y_4 = x_4 + y_3 = x_1 + x_3 + x_4.$$

After substituting the y_3 and y_4 expressions into the differential equations and writing them in matrix form, we obtain another state-space representation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 & 0 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -2 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}.$$

5. Consider a second-order system described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, such that its poles are located in the shaded region below.



Determine the smallest possible maximum percent-overshoot, the smallest possible peak time, and the smallest possible 2% settling-time of the system.

Solution: Maximum overshoot for a second-order system with no zero is given by

$$M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}.$$

The only system parameter that affects the maximum overshoot is ζ. For the smallest M_p, we need to have maximum ζ; since $\zeta = 0$ gives undamped oscillations. In the shaded region, the two boundaries of ζ are when $\zeta = \cos(30°) = \sqrt{3}/2$ and $\zeta = \cos(45°) = \sqrt{2}/2$. So, the maximum $\zeta = \sqrt{3}/2$, and the corresponding maximum overshoot is

$$M_p = e^{-\frac{\sqrt{3}/2}{\sqrt{1-(\sqrt{3}/2)^2}}\pi} = e^{-\sqrt{3}\pi} \approx 0.0043,$$

or the smallest possible maximum percent-overshoot is 0.43%.

The peak time of a second-order system with no zero is given by

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\sqrt{1-\zeta^2}\,\omega_n}.$$

The single system parameter that affects the settling time is ω_d. For the smallest t_p, we need to have maximum ω_d. In the shaded region, the maximum ω_d is at the intersection of the radial line with the 45° angle to the negative real axis and the circle with radius 20. Since the radial line with the 45° angle gives $\zeta = \cos(45°) = \sqrt{2}/2$, and the circle with radius 20 gives $\omega_n = 20$; we get

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \left(\sqrt{1-(\sqrt{2}/2)^2}\right)20 = \left(\sqrt{2}/2\right)20 = 10\sqrt{2}.$$

Therefore, the smallest possible peak time is $\pi/(10\sqrt{2})$ s or approximately 0.22 s.

The 2% settling time of a second-order system with no zero is given by

$$t_{2\%s} = \frac{4}{\sigma_o} = \frac{4}{\zeta\omega_n}.$$

The single system parameter that affects the settling time is σ_o. For the smallest $t_{2\%s}$, we need to have maximum σ_o. In the shaded region, the maximum σ_o is at the intersection of the radial line with the 30° angle to the negative real axis and the circle with radius 20. Since the radial line with the 30° angle gives $\zeta = \cos(30°) = \sqrt{3}/2$, and the circle with radius 20 gives $\omega_n = 20$; we get

$$\sigma_o = \zeta\omega_n = \left(\sqrt{3}/2\right) 20 = 10\sqrt{3}.$$

Therefore, the smallest possible 2% settling time is $4/(10\sqrt{3})$ s or approximately 0.23 s.

Note here that we can't choose all these variables at their smallest values, since they correspond to different pole locations.