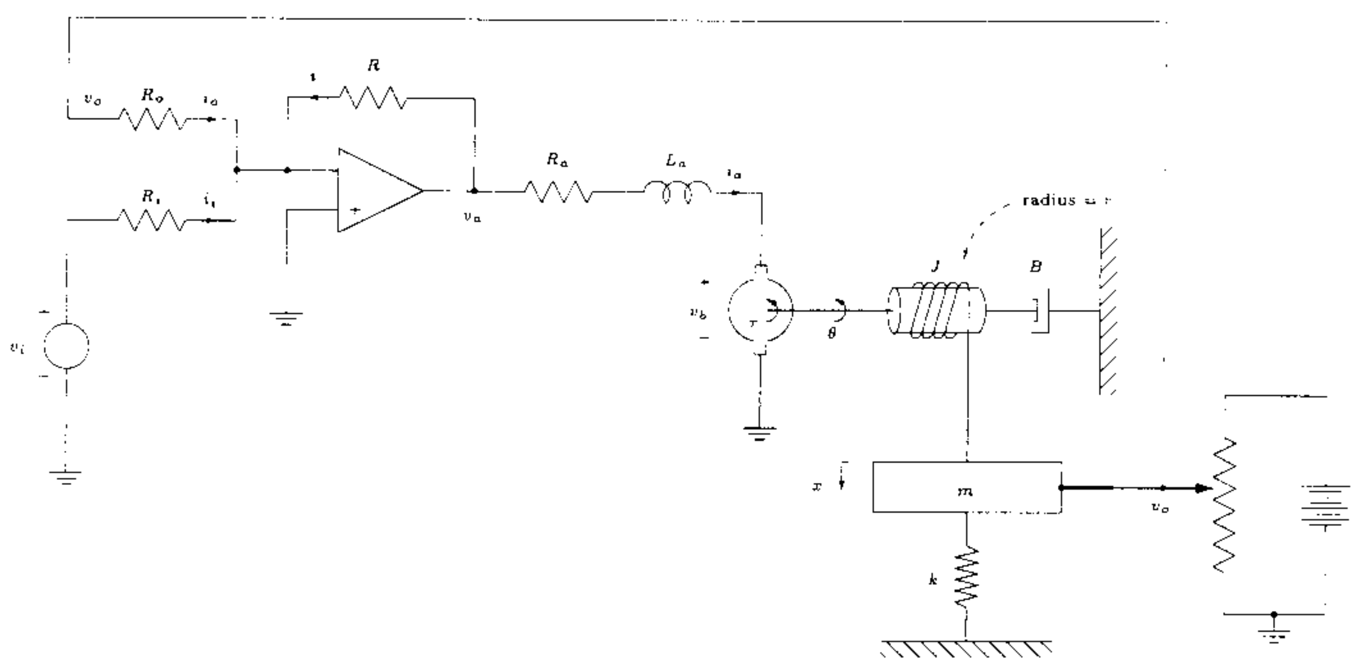
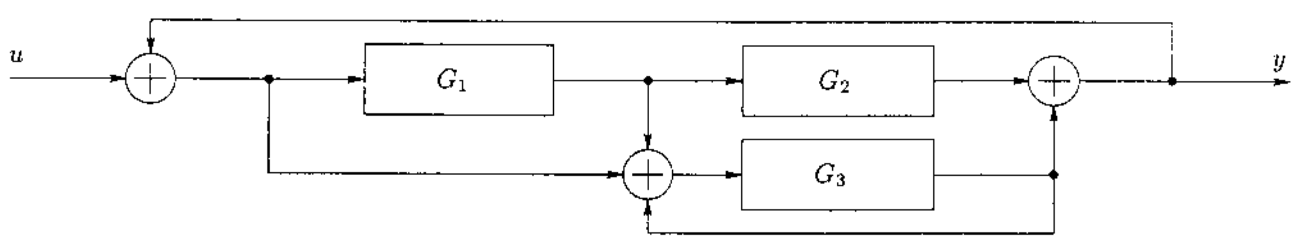
EE 231 **Exam#1** Mar. 03, 2000

50 minutes

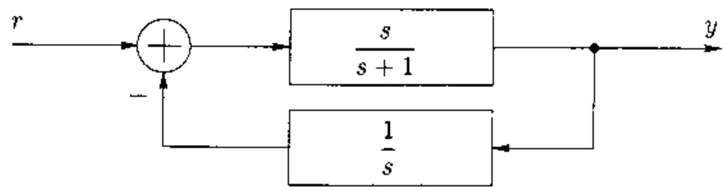
1. In the following system, an armature controlled motor is used to control the location of an object with mass m. At the output, a voltage that is proportional to the location of the object is generated such that $v_o = K_o x$. The displacement x of the object is measured so that $x = 0$, when $\theta = 0$. Obtain the detailed block diagram of the system, where v_i is the input and x is the output, and show the variables v_i, i_i, v_o, i_o, i, v_a, i_a, v_b, τ, θ, and x on the block diagram. Assume that the spring is compressed initially to compensate for the force due to gravity. (30pts)



2. For the block diagram given below, determine the transfer function *either* by block diagram reduction, *or* by Mason's formula. Show your work clearly. (25pts)

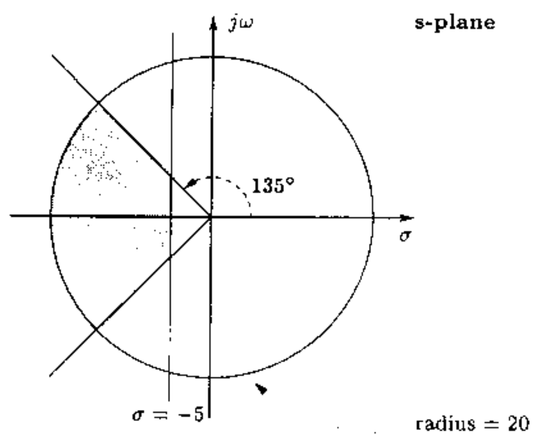


3. The block diagram of a control system is given below.



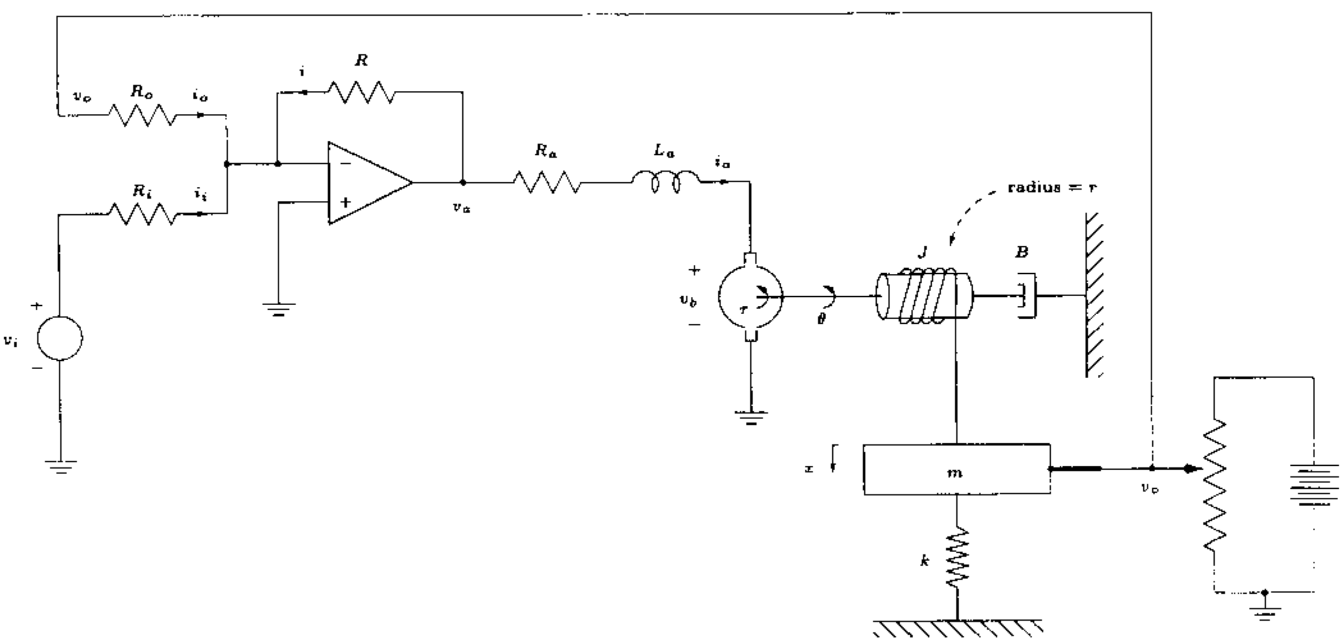
Obtain a state-space representation of the system without any block-diagram reduction. (25pts)

4. Obtain the necessary inequalities to describe the poles in the shaded region below in terms of only ζ and ω_n of a second-order system with complex poles described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$. (20pts)

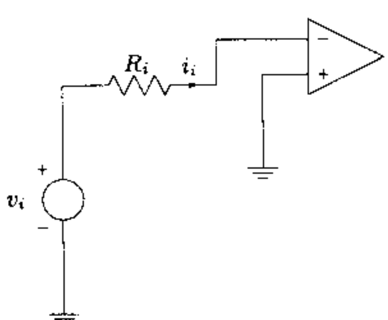


EE 231 **Exam#1 Solutions** Mar. 03, 2000

1. In the following system, an armature controlled motor is used to control the location of an object with mass m. At the output, a voltage that is proportional to the location of the object is generated such that $v_o = K_o x$. The displacement x of the object is measured so that $x = 0$, when $\theta = 0$. Obtain the detailed block diagram of the system, where v_i is the input and x is the output, and show the variables v_i, i_i, v_o, i_o, i, v_a, i_a, v_b, τ, θ, and x on the block diagram. Assume that the spring is compressed initially to compensate for the force due to gravity.



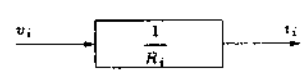
Solution: To determine the block diagram of the system, we first separate it into simpler components.

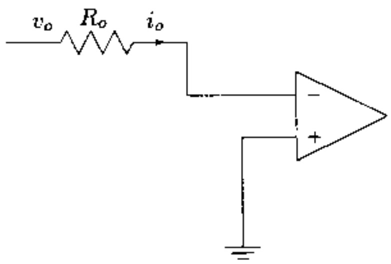


Because the input variable is v_i, we write i_i in terms v_i, such that

$$I_i = \frac{1}{R_i} V_i,$$

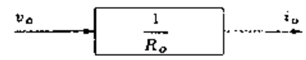
since the operational amplifier is assumed to be ideal.

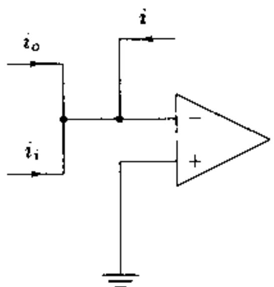




Similarly, we have

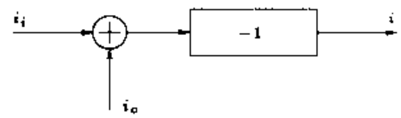
$$I_o = \frac{1}{R_o} V_o.$$

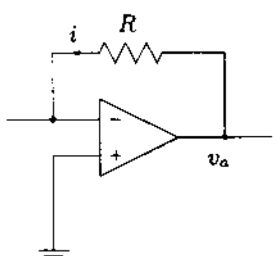




For an ideal operational amplifier,

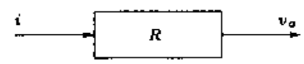
$$i = -(i_i + i_o).$$

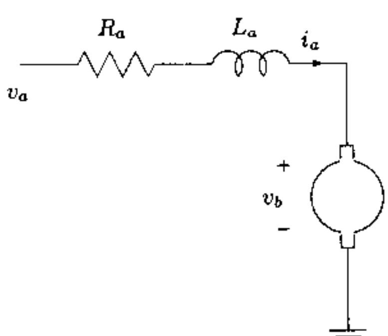




Again for an ideal operational amplifier,

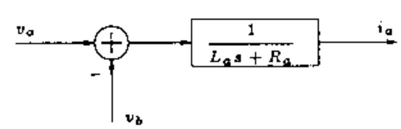
$$V_a = RI.$$

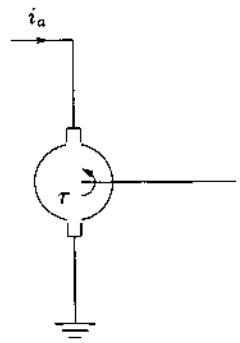




The armature current of the motor is

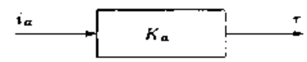
$$I_a = \frac{1}{L_a s + R_a}(V_a - V_b).$$

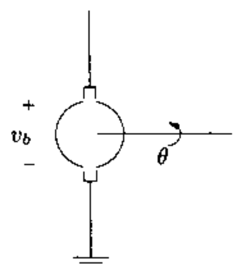




From the armature controlled motor,

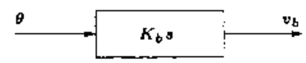
$$\tau = K_a i_a.$$

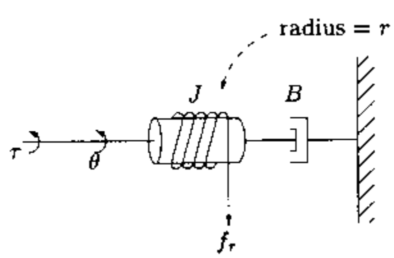




The back-emf voltage of the motor

$$V_b = (K_b s)\Theta.$$



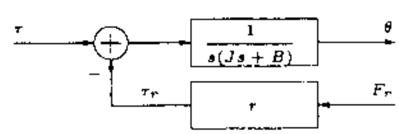


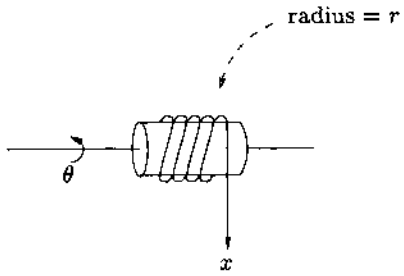
The torque equation

$$J\ddot{\theta} = \tau - B\dot{\theta} - \tau_r,$$

where τ_r is the torque generated due to the spring-mass system, such that $\tau_r = r f_r$, and f_r is the rope tension on the free-body diagram. So,

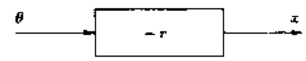
$$\Theta = \frac{1}{s(Js + B)}(T - rF_r).$$

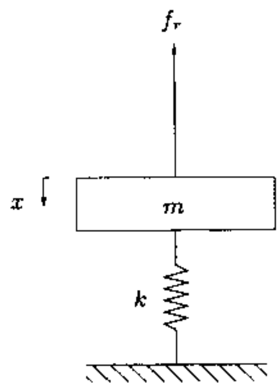




The disc with the inertia J changes the rotational motion to translational motion, where

$$x = -r\theta.$$



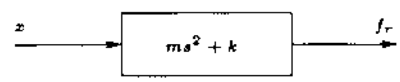


From the free body diagram, we get one equation describing the translational motion.

$$m\ddot{x} = -f_r - kx,$$

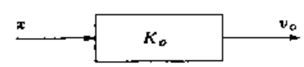
resulting in

$$F_r = (ms^2 + k)X.$$

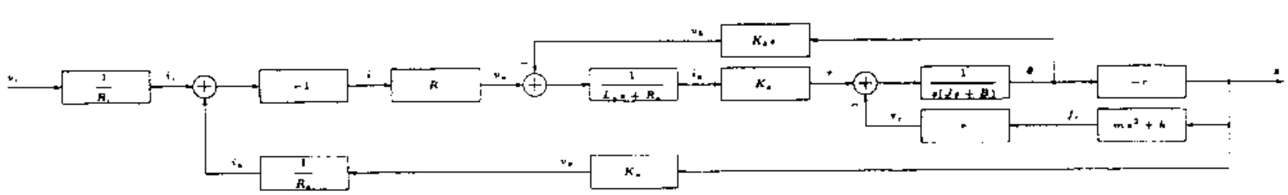


And, finally the given relationship

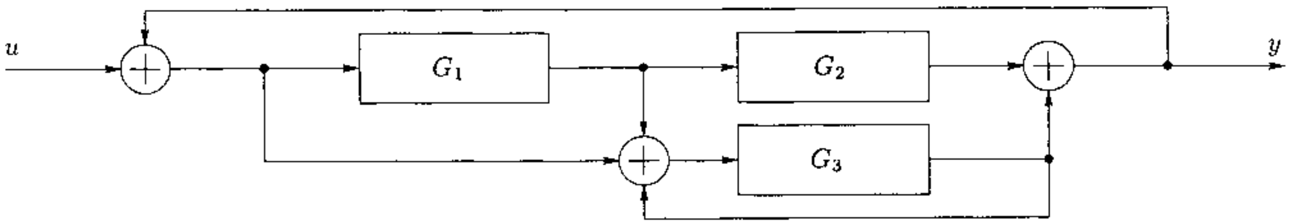
$$v_o = K_o x.$$



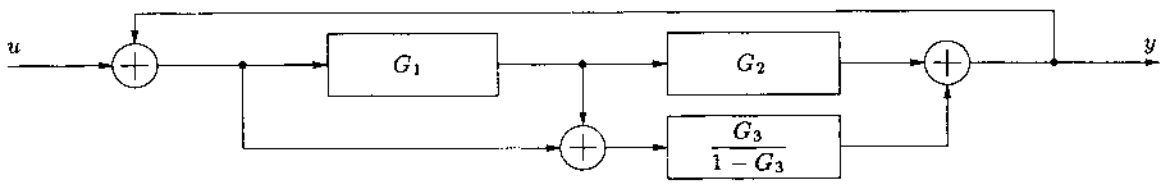
When we connect all the individual blocks together, we get the following block diagram.

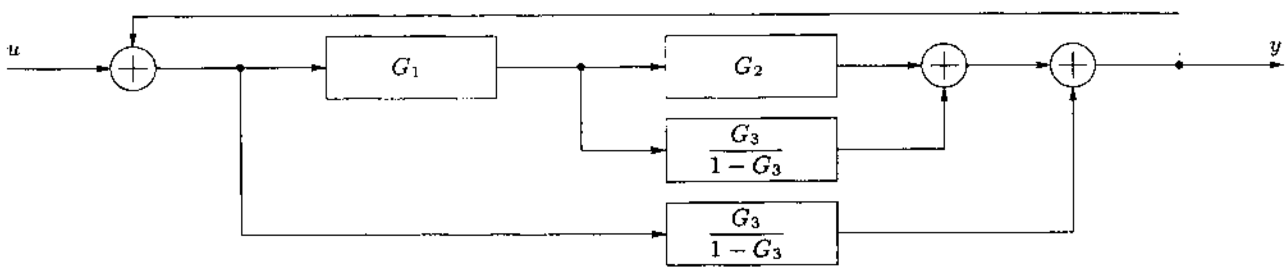


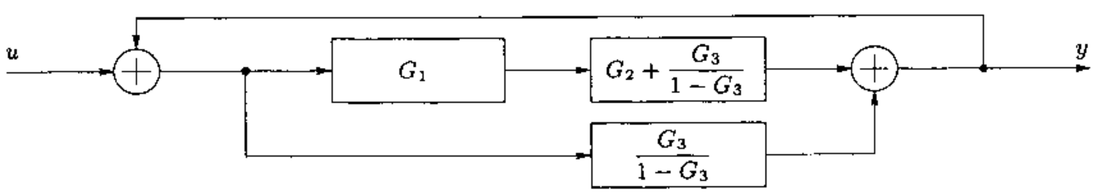
2. For the block diagram given below, determine the transfer function *either* by block diagram reduction, *or* by Mason's formula. Show your work clearly.

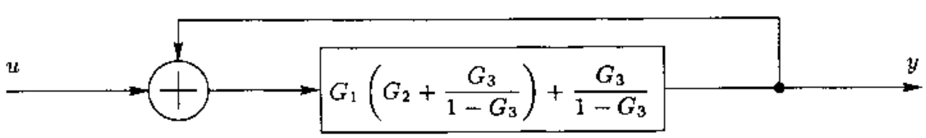


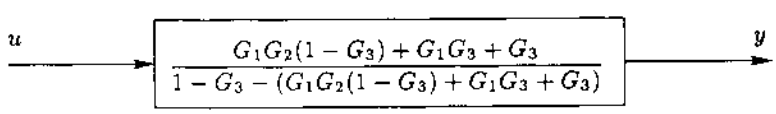
Solution: If we choose to use the block diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.



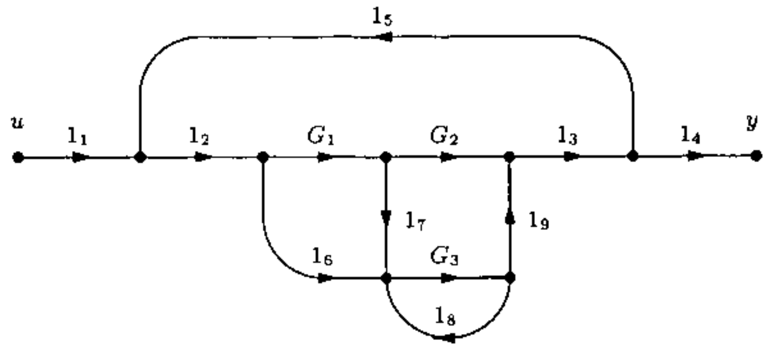








If we choose to use Mason's formula, we need to draw the signal flow graph of the block diagram.



In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The forward path gains are

$$F_1 = 1_1 1_2 G_1 G_2 1_3 1_4 = G_1 G_2,$$
$$F_2 = 1_1 1_2 1_6 G_3 1_9 1_3 1_4 = G_3,$$

and

$$F_3 = 1_1 1_2 G_1 1_7 G_3 1_9 1_3 1_4 = G_1 G_3.$$

The loop gains are

$$L_1 = G_3 1_8 = G_3,$$
$$L_2 = 1_2 G_1 G_2 1_3 1_5 = G_1 G_2,$$
$$L_3 = 1_2 1_6 G_3 1_9 1_3 1_5 = G_3,$$

and

$$L_4 = 1_2 G_1 1_7 G_3 1_9 1_3 1_5 = G_1 G_3.$$

From the forward path and the loop gains, we determine the touching loops and the forward paths.

Touching Loops

	L_1	L_2	L_3	L_4
L_1	✔	✘	✔	✔
L_2		✔	✔	✔
L_3			✔	✔
L_4				✔

Loops on Forward Paths

	L_1	L_2	L_3	L_4
F_1	✘	✔	✔	✔
F_2	✔	✔	✔	✔
F_3	✔	✔	✔	✔

Therefore,

$$\begin{aligned}\Delta &= 1 - \big(L_1 + L_2 + L_3 + L_4\big) + \big(L_1 L_2\big) \\ &= 1 - \big((G_3) + (G_1 G_2) + (G_3) + (G_1 G_3)\big) + \big((G_3)(G_1 G_2)\big) \\ &= 1 - 2G_3 - G_1 G_2 - G_1 G_3 + G_1 G_2 G_3,\end{aligned}$$

and

$$\begin{aligned}\Delta_1 &= \Delta|_{L_2=L_3=L_4=0} = 1 - L_1 = 1 - G_3, \\ \Delta_2 &= \Delta|_{L_1=L_2=L_3=L_4=0} = 1, \\ \Delta_3 &= \Delta|_{L_1=L_2=L_3=L_4=0} = 1.\end{aligned}$$

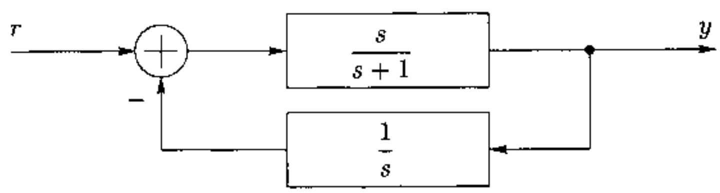
So,

$$\frac{Y(s)}{U(s)} = \frac{1}{\Delta}\sum_{i=1}^{3} F_i \Delta_i = \frac{(G_1G_2)(1-G_3) + (G_3)(1) + (G_1G_3)(1)}{1 - 2G_3 - G_1G_2 - G_1G_3 + G_1G_2G_3},$$

or

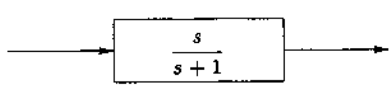
$$\frac{Y(s)}{U(s)} = \frac{G_1G_2 - G_1G_2G_3 + G_3 + G_1G_3}{1 - 2G_3 - G_1G_2 - G_1G_3 + G_1G_2G_3}.$$

3. The block diagram of a control system is given below.

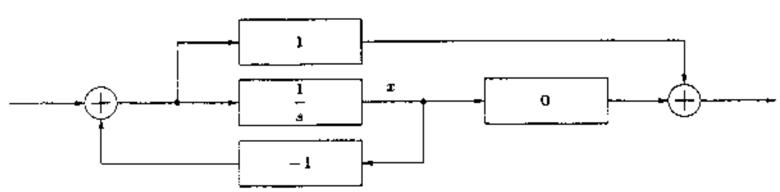


Obtain a state-space representation of the system without any block-diagram reduction.

Solution: In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.

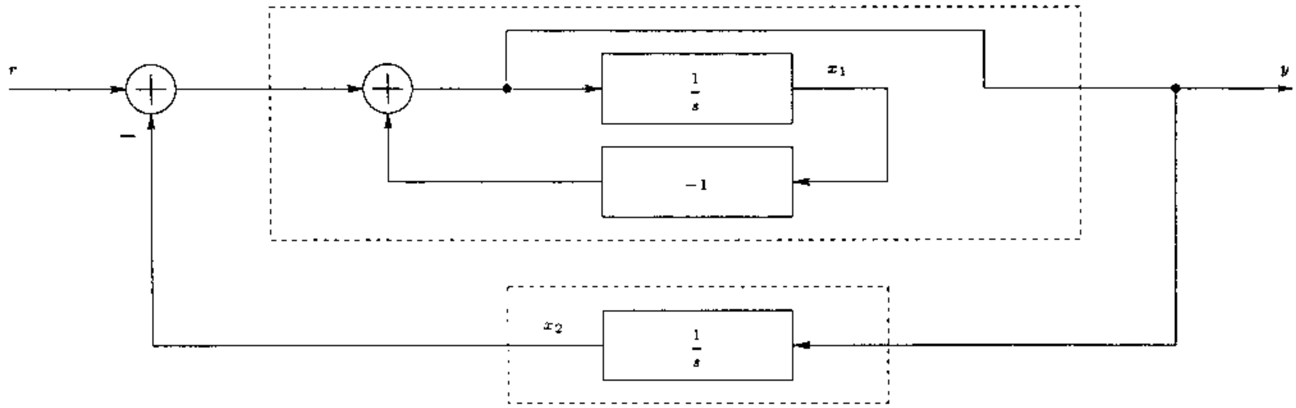


(a) The feedforward gain block.



(b) Controller realization form.

There is no need to generate a realization for the feedback gain block, since it is already in a realization form. The connected and "expanded" block diagram is shown below.



After assigning the state variables as shown in the figure, we obtain

$$\dot{x}_1 = -x_1 + (r - x_2) = -x_1 - x_2 + r,$$
$$\dot{x}_2 = \dot{x}_1 = -x_1 - x_2 + r,$$

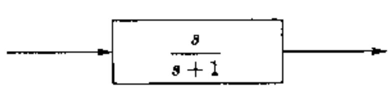
and

$$y = \dot{x}_1 = -x_1 - x_2 + r.$$

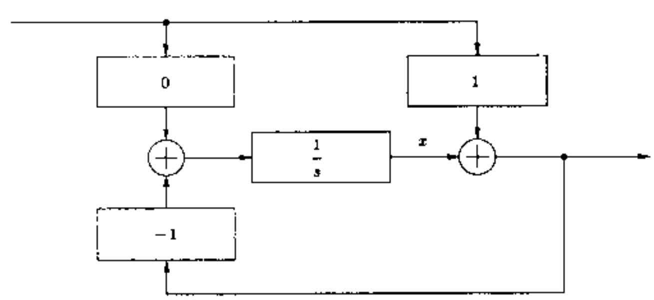
And the state-space representation is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} r(t),$$
$$y(t) = \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} r(t).$$

If we use the observer realization form for each of the blocks, then we obtain a different state-space representation.

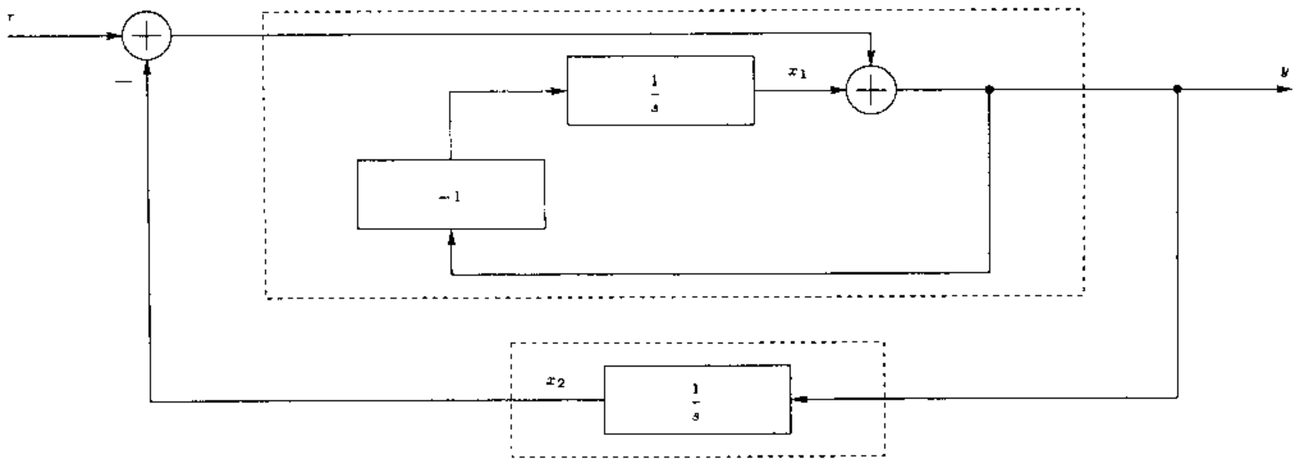


(a) The feedback gain block.



(b) Observer realization form.

The connected and "expanded" block diagram for this case is shown below.



Similarly, we obtain

$$\dot{x}_1 = -y = -(x_1 + (r - x_2)) = -x_1 + x_2 - r,$$
$$\dot{x}_2 = y = x_1 + (r - x_2) = x_1 - x_2 + r,$$

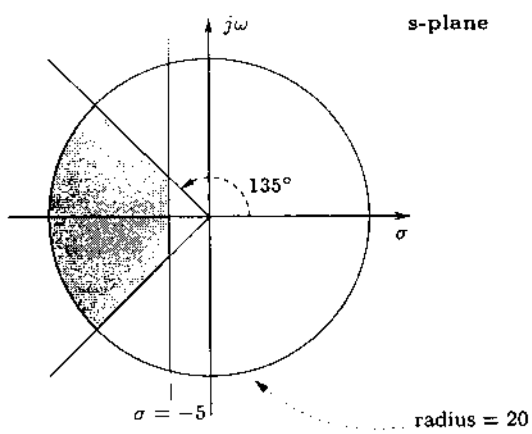
and

$$y = x_1 - x_2 + r.$$

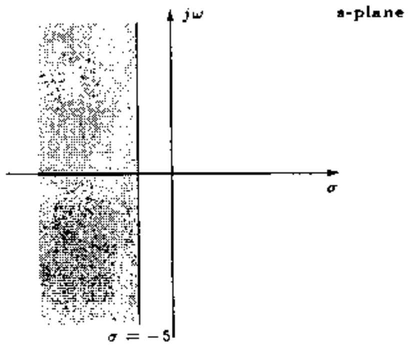
And,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} r(t),$$
$$y(t) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [1] r(t).$$

4. Obtain the necessary inequalities to describe the poles in the shaded region below in terms of only ζ and ω_n of a second-order system with complex poles described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$.



Solution: To be able to describe the shaded region, we need to separate it into unions or intersections of simpler regions.

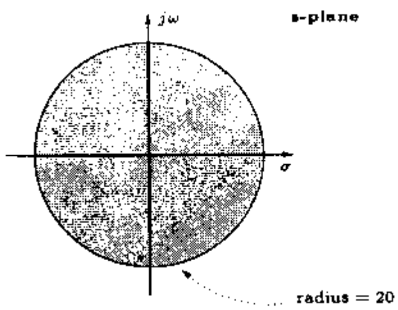


A vertical straight line designates a constant value for the real part of the poles. Since the real part of the complex poles are at $-\zeta\omega_n$, the shown shaded area is represented by

$$-\zeta\omega_n \leq -5,$$

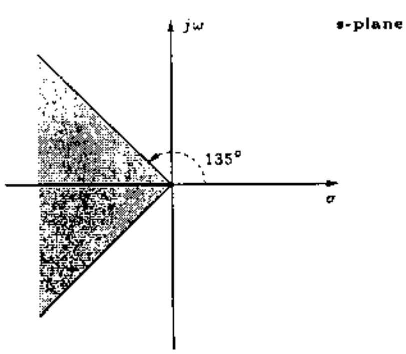
or

$$\zeta\omega_n \geq 5.$$



The equi-distance points from the origin designate constant value for ω_n. As a result, the shown shaded area is represented by

$$\omega_n \leq 20.$$



A straight line originating from the origin designates a constant ζ value, where $\cos^{-1}(\zeta)$ is the acute angle between the line and the negative real axis. So for the shaded area shown, we have

$$0^\circ < \cos^{-1}(\zeta) \leq 180^\circ - 135^\circ,$$

or

$$\cos(0^\circ) > \zeta \geq \cos(45^\circ),$$

since $\cos(\theta)$ is a monotonically decreasing function for $0 < \theta < 180^\circ$. So, we have

$$\frac{\sqrt{2}}{2} \leq \zeta < 1,$$

when the poles have non-zero imaginary parts.

Therefore, the shaded area given in the problem is the intersection of the individual shaded areas, and it can be represented by

$$\zeta\omega_n \geq 5,$$

$$\omega_n \leq 20,$$

$$\sqrt{2}/2 \leq \zeta < 1.$$