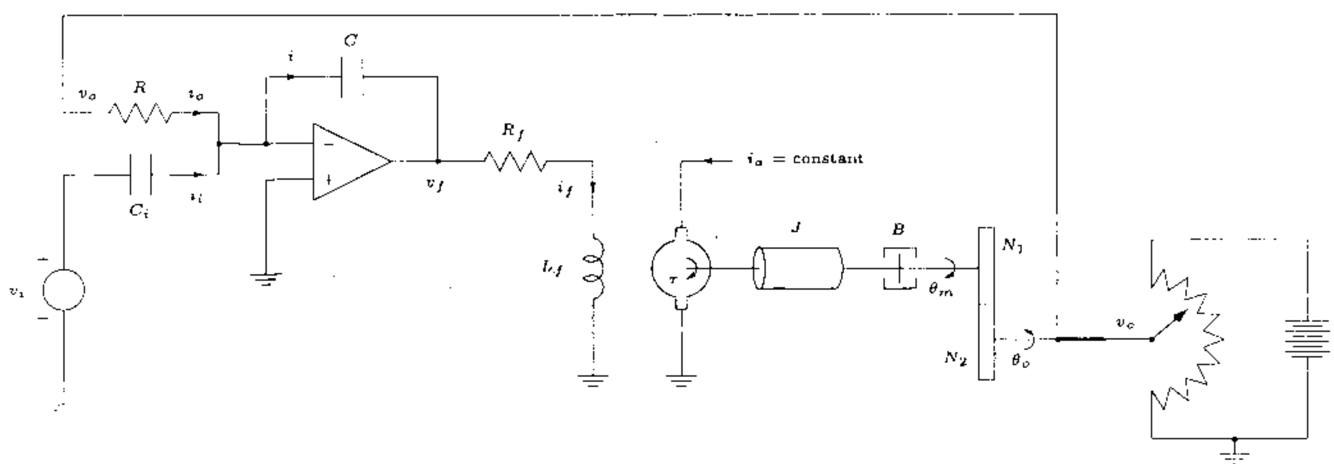
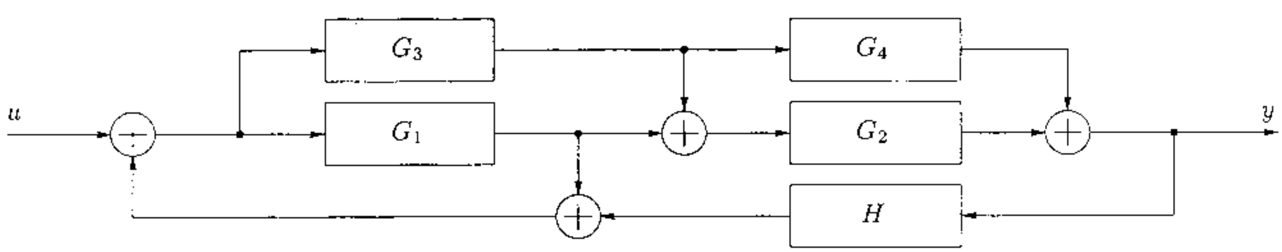
EE 231 | **Exam#1** 50 minutes | **Mar. 09, 2001**

1. The angular position of the shaft of a motor is controlled by the system shown below.

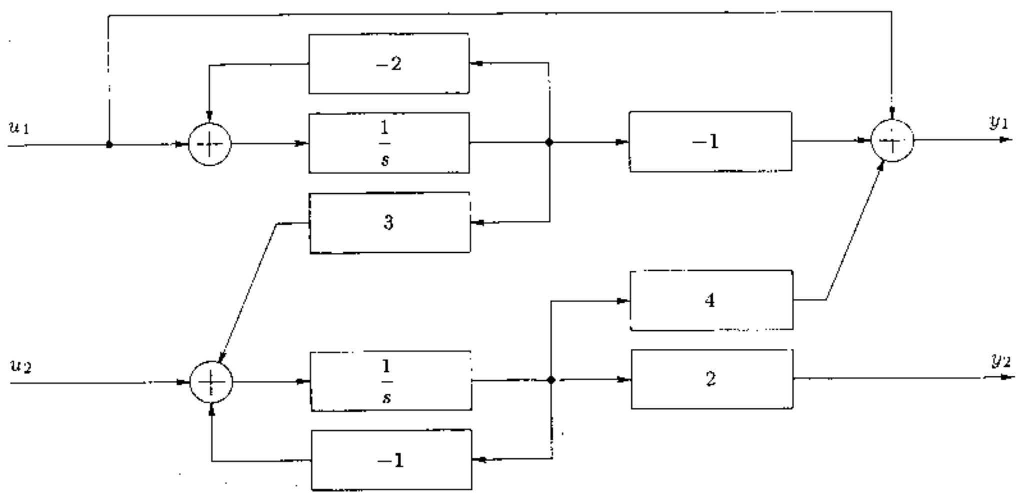


The angular position of the motor shaft is detected by a variable resistor which provides a voltage v_o proportional to the angle, such that $v_o = -k_o\theta_o$. Draw the most detailed block diagram of the system, where v_i is the input, and θ_o is the output. Show all the variables v_i, i_i, v_o, i_o, i, v_f, i_f, τ, θ_m, and θ_o on the block diagram. (30pts)

2. For the block diagram given below, determine the transfer function *either* by block diagram reduction, *or* by Mason's formula. Show your work clearly. (25pts)



3. Consider the following block diagram representation of a control system, determine the transfer matrix of the system. (25pts)



4. A control system is represented by

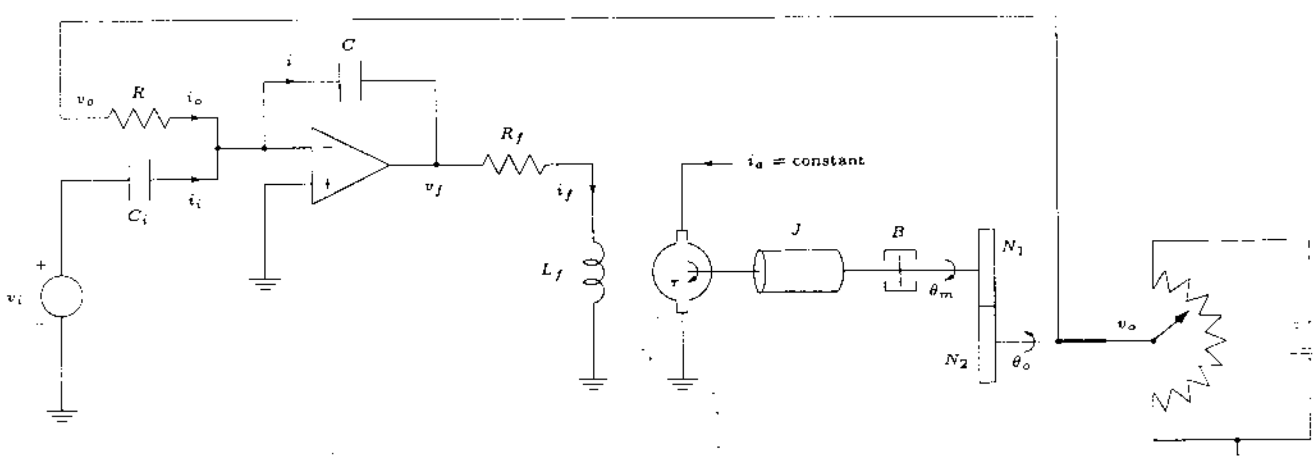
$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \end{bmatrix} u(t).$$

Determine $y(t)$ for $t \geq 0$; when $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, and $u(t) = 0$ for $t \geq 0$. (20pts)

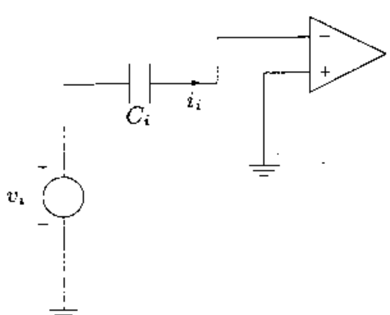
EE 231 | Exam#1 Solutions | Mar. 09, 2001

1. The angular position of the shaft of a motor is controlled by the system shown below.



The angular position of the motor shaft is detected by a variable resistor which provides a voltage v_o proportional to the angle, such that $v_o = K_o\theta_o$. Draw the most detailed block diagram of the system, where v_i is the input, and θ_o is the output. Show all the variables v_i, i_i, v_o, i_o, i, v_f, i_f, τ, θ_m, and θ_o on the block diagram.

Solution: To determine the block diagram of the system, we first separate it into simpler components.



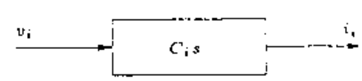
Because the input variable is v_i, we write i_i in terms v_i, such that

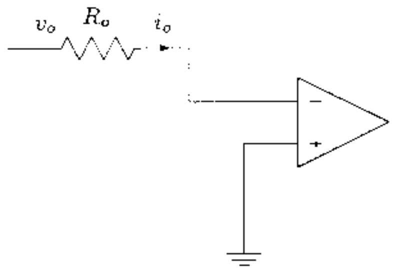
$$i_i(t) = C_i\frac{\mathrm{d}v_i(t)}{\mathrm{d}t},$$

or

$$I_i(s) = C_i s V_i(s),$$

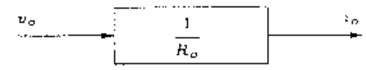
since the operational amplifier is assumed to be ideal.

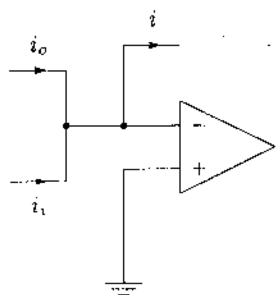




Similarly, we have

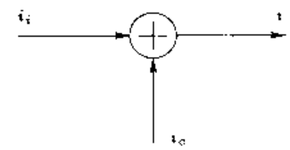
$$I_o(s) = \frac{1}{R_o} V_o(s).$$

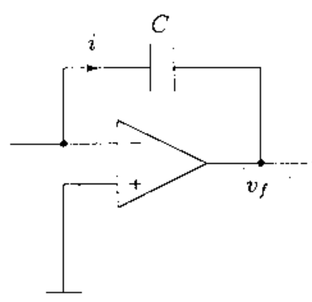




For an ideal operational amplifier,

$$i(t) = i_i(t) + i_o(t).$$



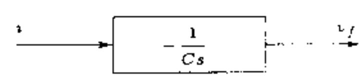


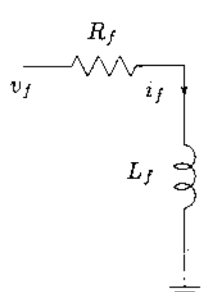
Again for an ideal operational amplifier,

$$v_f(t) = -\frac{1}{C}\int^t i(\xi)\,\mathrm{d}\xi,$$

or

$$V_f(s) = -\frac{1}{Cs} I(s).$$



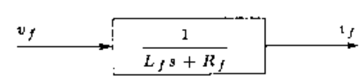


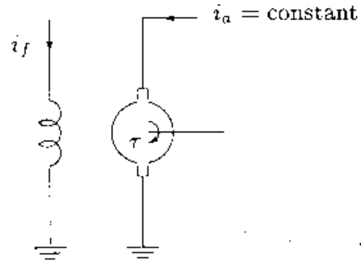
The field current of the motor can be obtained from the Kirchhoff's Voltage Law, where

$$L_f \frac{\mathrm{d}i_f(t)}{\mathrm{d}t} + R_f i_f(t) = v_f(t),$$

or

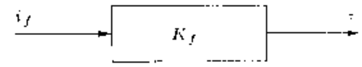
$$I_f(s) = \frac{1}{L_f s + R_f} V_f(s).$$

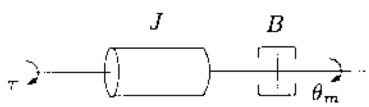




From the field controlled motor,

$$\tau(t) = K_f i_f(t).$$



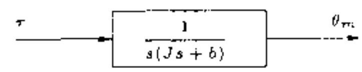


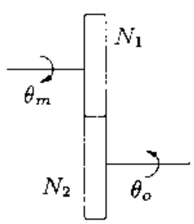
The torque equation is

$$J\frac{\mathrm{d}^2\theta_m(t)}{\mathrm{d}t^2} = \tau(t) - B\frac{\mathrm{d}\theta(t)}{\mathrm{d}t},$$

or

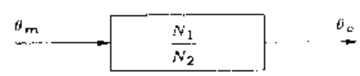
$$\Theta_m(s) = \frac{1}{s(Js+B)}T(s).$$





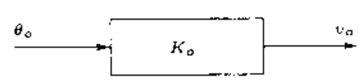
The gear relates the two angle values, such that

$$\theta_o(t) = \frac{N_1}{N_2}\theta_m(t).$$

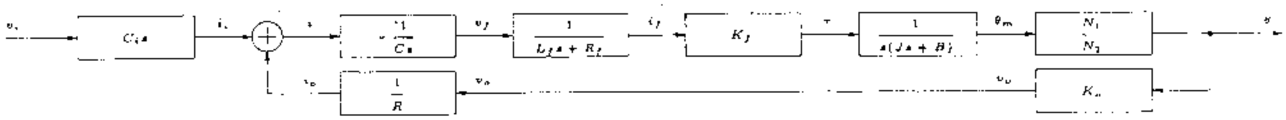


And, finally the given relationship

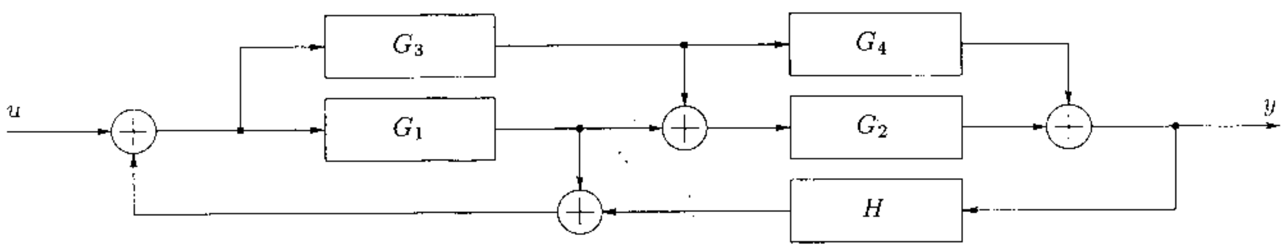
$$v_o(t) = K_o\theta_o(t).$$



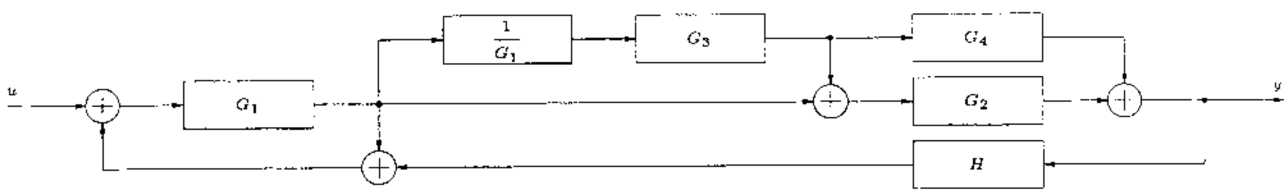
When we connect all the individual blocks together, we get the following block diagram.

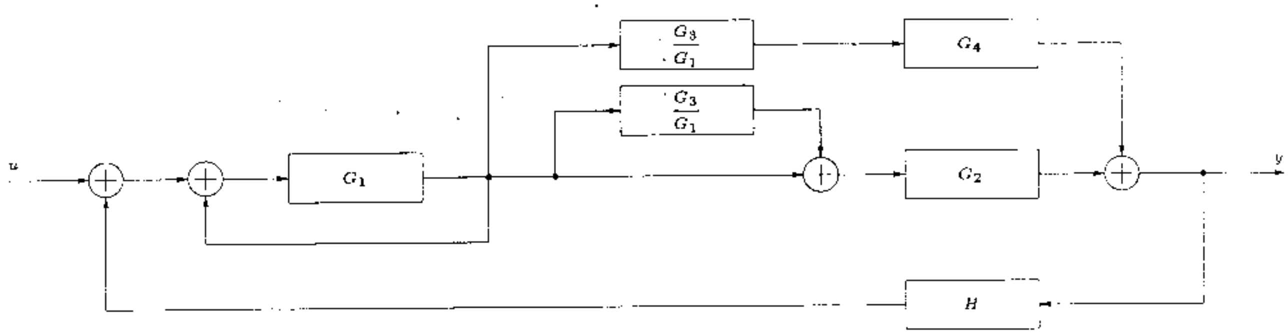


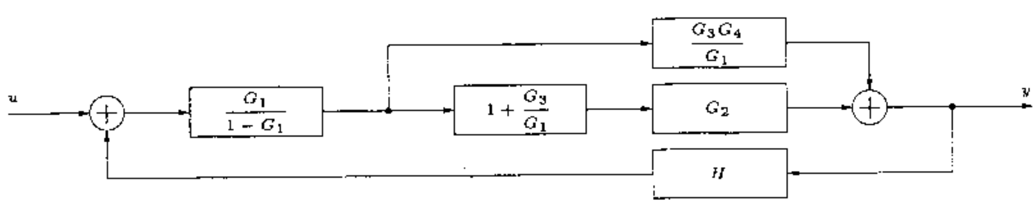
2. For the block diagram given below, determine the transfer function *either* by block diagram reduction. *or* by Mason's formula. Show your work clearly.

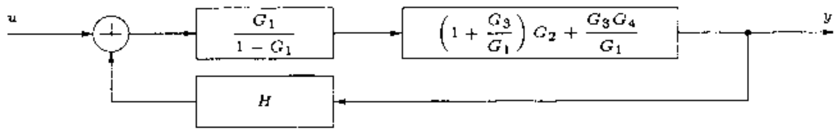


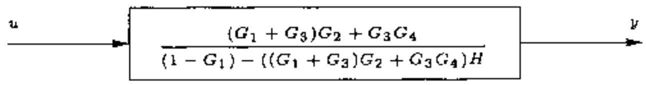
Solution: If we choose to use the block diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.



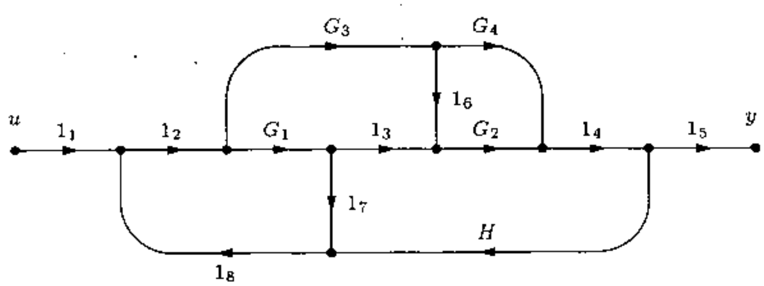








If we choose to use Mason's formula, we need to draw the signal flow graph of the block diagram.



In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The forward path gains are

$$F_1 = 1_1 1_2 G_1 1_3 G_2 1_4 1_5 = G_1 G_2,$$
$$F_2 = 1_1 1_2 G_3 1_6 G_2 1_4 1_5 = G_2 G_3,$$

and

$$F_3 = 1_1 1_2 G_3 G_4 1_4 1_5 = G_3 G_4.$$

The loop gains are

$$L_1 = 1_2 G_1 1_7 1_8 = G_1,$$
$$L_2 = 1_2 G_1 1_3 G_2 1_4 H 1_8 = G_1 G_2 H,$$
$$L_3 = 1_2 G_3 1_6 G_2 1_4 H 1_8 = G_2 G_3 H,$$

and

$$L_4 = 1_2 G_3 G_4 1_4 H 1_8 = G_3 G_4 H.$$

From the forward path and the loop gains, we determine the touching loops and the forward paths.

Touching Loops

	L_1	L_2	L_3	L_4
L_1	✔	✔	✔	✔
L_2		✔	✔	✔
L_3			✔	✔
L_4				✔

Loops on Forward Paths

	L_1	L_2	L_3	L_4
F_1	✔	✔	✔	✔
F_2	✔	✔	✔	✔
F_3	✔	✔	✔	✔

Therefore,

$$\begin{aligned}\Delta &= 1 - \big(L_1 + L_2 + L_3 + L_4\big) \\ &= 1 - \big((G_1) + (G_1G_2H) + (G_2G_3H) + (G_3G_4H)\big) \\ &= 1 - G_1 - G_1G_2H - G_2G_3H - G_3G_4H,\end{aligned}$$

and

$$\begin{aligned}\Delta_1 &= \Delta|_{L_1=L_2=L_3=L_4=0} = 1, \\ \Delta_2 &= \Delta|_{L_1=L_2=L_3=L_4=0} = 1, \\ \Delta_3 &= \Delta|_{L_1=L_2=L_3=L_4=0} = 1.\end{aligned}$$

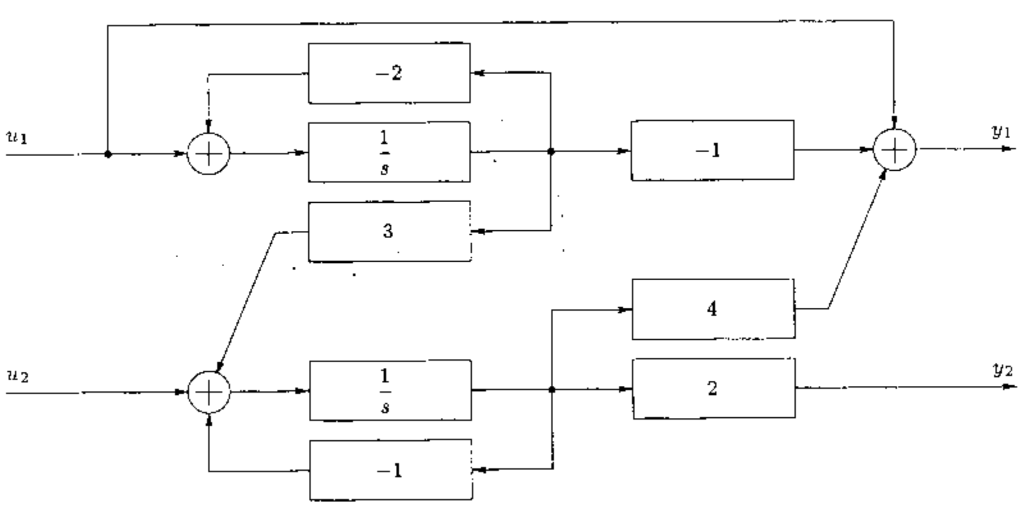
So,

$$\frac{Y(s)}{U(s)} = \frac{1}{\Delta}\sum_{i=1}^{3} F_i\Delta_i = \frac{(G_1G_2)(1) + (G_2G_3)(1) + (G_3G_4)(1)}{1 - G_1 - G_1G_2H - G_2G_3H - G_3G_4H},$$

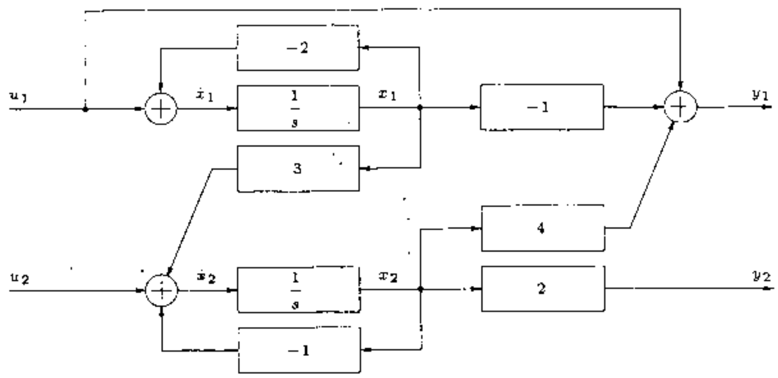
or

$$\frac{Y(s)}{U(s)} = \frac{G_1G_2 + G_2G_3 + G_3G_4}{1 - G_1 - G_1G_2H - G_2G_3H - G_3G_4H}.$$

3. Consider the following block diagram representation of a control system, determine the transfer matrix of the system.



Solution: The block diagram is in a realization form, so we can assign state variables at the outputs of the integrators and obtain the state-space equations from the realization.



$$\dot{x}_1 = -2x_1 + u_1,$$
$$\dot{x}_2 = 3x_1 - x_2 + u_2,$$

and

$$y_1 = -x_1 + 4x_2 + u_1,$$
$$y_2 = 2x_2.$$

In vector form,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$

and

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

The transfer matrix of a system given in state-space representation is

$$F(s) = C(sI - A)^{-1}B + D,$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} -1 & 4 \\ 0 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and I is the appropriately dimensioned identity matrix. So,

$$F(s) = \begin{bmatrix} -1 & 4 \\ 0 & 2 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 4 \\ 0 & 2 \end{bmatrix} \left(\begin{bmatrix} s+2 & 0 \\ -3 & s+1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{(s+2)(s+1) - (-3)(0)} \begin{bmatrix} -1 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 3 & s+2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)} \begin{bmatrix} -s+11 & 4s+8 \\ 6 & 2s+4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-s+11}{(s+1)(s+2)} + 1 & \frac{4(s+2)}{(s+1)(s+2)} \\ \frac{6}{(s+1)(s+2)} & \frac{2(s+2)}{(s+1)(s+2)} \end{bmatrix}.$$

Therefore, the transfer matrix is

$$F(s) = \begin{bmatrix} \frac{s^2+2s+13}{(s+1)(s+2)} & \frac{4}{(s+1)} \\ \frac{6}{(s+1)(s+2)} & \frac{2}{(s+1)} \end{bmatrix}.$$

4. A control system is represented by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \end{bmatrix} u(t).$$

Determine $y(t)$ for $t \geq 0$; when $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, and $u(t) = 0$ for $t \geq 0$.

Solution: The general solution to the state-space representation of a system described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t)$$

is obtained from

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)} B\mathbf{u}(\tau)\, d\tau,$$

where

$$e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right].$$

Here, I is the appropriately dimensioned identity matrix. In our case,

$$A = \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 \end{bmatrix},$$

and $u(t) = 0$ for $t \geq 0$. As a result, the integral term in the solution of x and the second term in the y equation are identically zero. So,

$$\begin{aligned}
y(t) &= Ce^{At}\mathbf{x}(0) \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathcal{L}^{-1}\left[\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix}\right)^{-1}\right] \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathcal{L}^{-1}\left[\left(\begin{bmatrix} s+1 & 0 \\ -4 & s+2 \end{bmatrix}\right)^{-1}\right] \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2) - (-4)(0)} \begin{bmatrix} s+2 & 0 \\ 4 & s+1 \end{bmatrix}\right] \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] & 0 \\ \mathcal{L}^{-1}\left[\frac{4}{(s+1)(s+2)}\right] & \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

Or,

$$y(t) = 0 \qquad \text{for } t \geq 0.$$