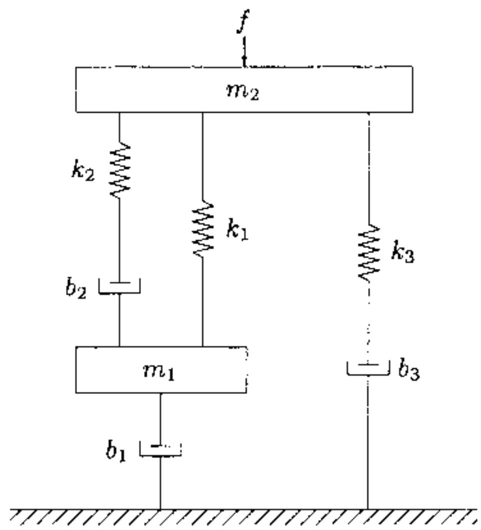
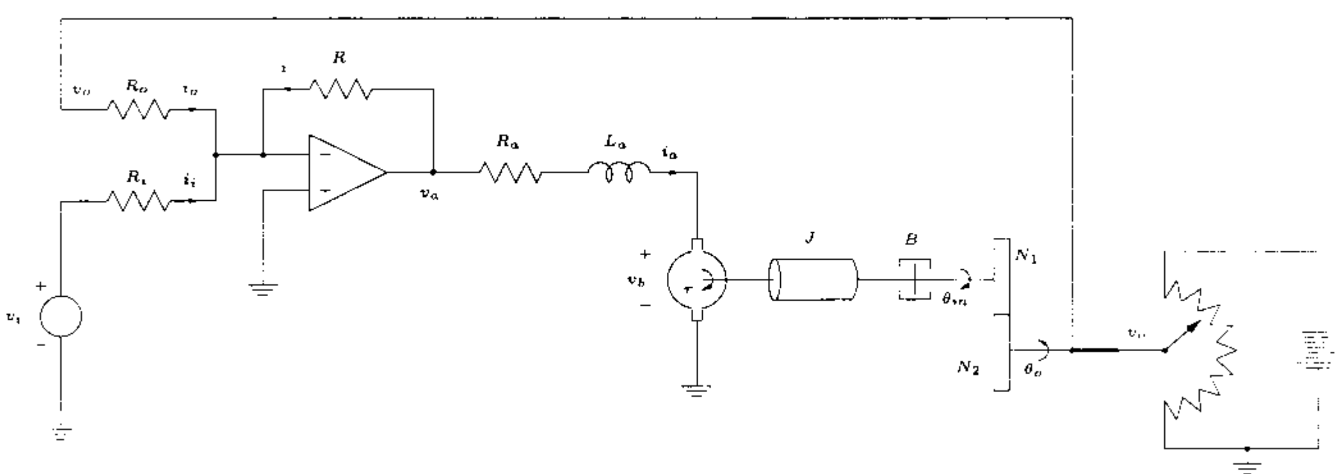
75 minutes

1. For the mechanical system shown below, obtain *either* the force-voltage *or* the force-current analog of the system. (25pts)

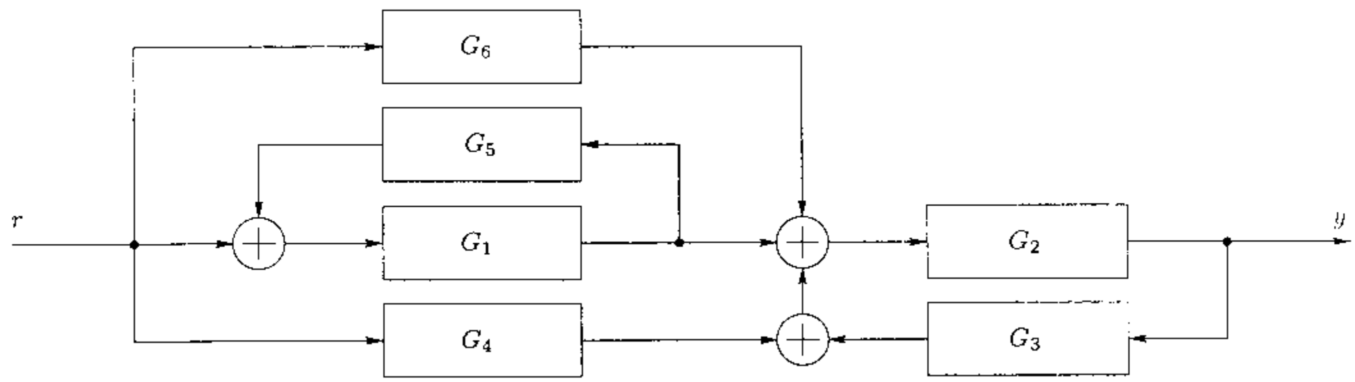


2. The angular position of the shaft of a motor is controlled by the system shown below.

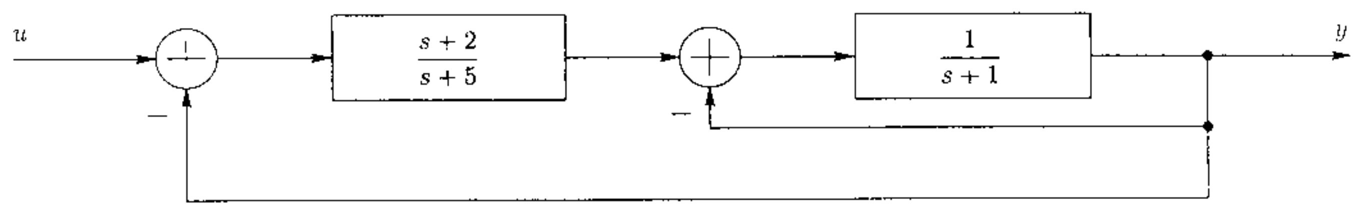


The angular position of the motor shaft is detected by a variable resistor which provides a voltage v_o proportional to the angle, such that $v_o = K_o\theta_o$. Draw the most detailed block diagram of the system, where v_i is the input, and θ_o is the output. Show all the variables v_i, i_i, v_o, i_o, i, v_a, i_a, v_b, τ, θ_m, and θ_o on the block diagram. (25pts)

3. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly. (25pts)



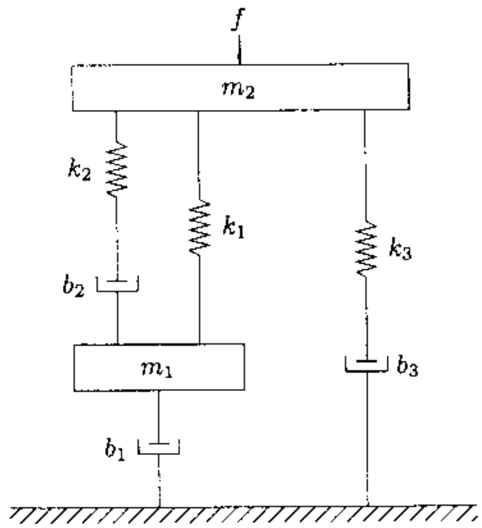
4. The block diagram of a control system is given below.



(a) Obtain a state-space representation of the system without any block-diagram reduction. (20pts)

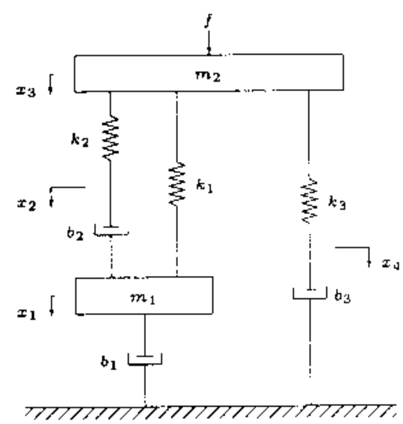
(b) Determine the transfer function of the system. (05pts)

1. For the mechanical system shown below, obtain *either* the force-voltage *or* the force-current analog of the system.



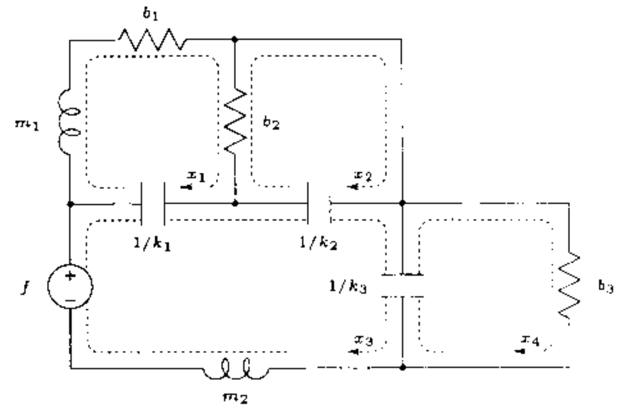
Solution:

We first identify the linearly independent displacement locations in the mechanical system and mark them.



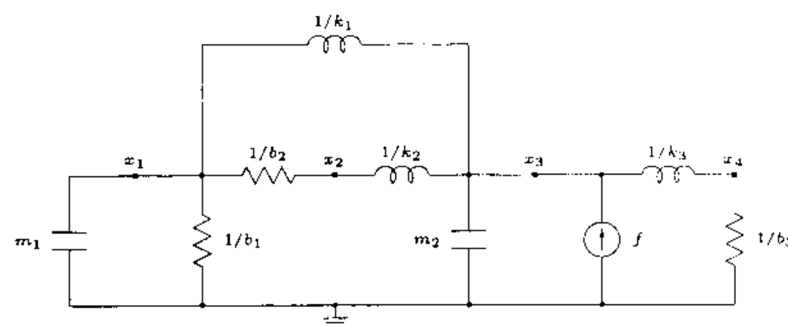
For the force-voltage analog of a mechanical system, there will be a loop charge associated with each displacement variable (or a loop current associated with each velocity variable), and an input force will be associated with a voltage source. The spring constant, the damping constant, and the mass will be associated with the reciprocal of capacitance, the resistance, and the inductance, respectively. The elements between two displacement variables of the mechanical system will be between the corresponding loop variables of the force-voltage analog. The elements that are connected to fixed frames and the elements that are always measured with respect to a fixed frame, such as the mass and the external force, will be on the non-common portions of the loops.

The next figure shows the force-voltage analog of the mechanical system, where the loops are identified with the displacements.

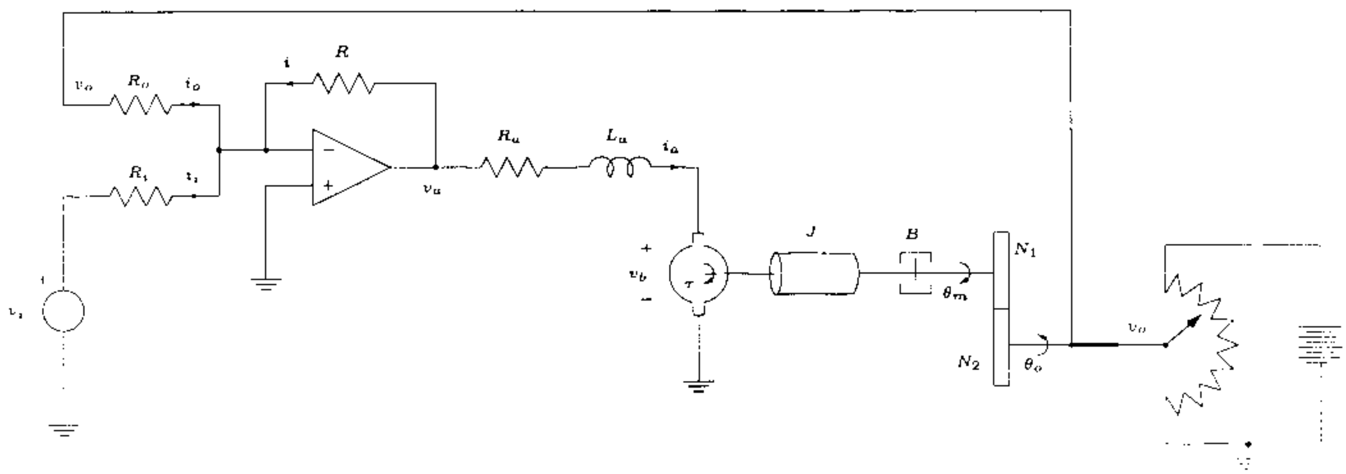


For the force-current analog of a mechanical system, there will be a node flux associated with each displacement variable (or a node voltage associated with each velocity variable), and an input force will be associated with a current source. The spring constant, the damping constant, and the mass will be associated with the reciprocal of inductance, the conductance, and the capacitance, respectively. The elements between two displacement variables of the mechanical system will be between the corresponding node variables of the force-voltage analog. The elements that are connected to fixed frames and the elements that are always measured with respect to a fixed frame, such as the mass and the external force, will be connected to the ground.

The next figure shows the force-current analog of the mechanical system, where the nodes are identified with the displacements.

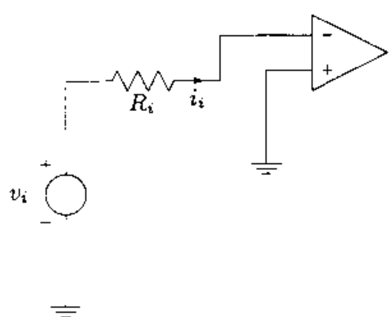


2. The angular position of the shaft of a motor is controlled by the system shown below.



The angular position of the motor shaft is detected by a variable resistor which provides a voltage v_o proportional to the angle, such that $v_o = K_o\theta_o$. Draw the most detailed block diagram of the system, where v_i is the input, and θ_o is the output. Show all the variables v_i, i_i, v_o, i_o, i, v_a, i_a, v_b, τ, θ_m, and θ_o on the block diagram.

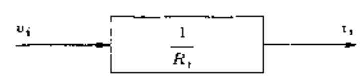
Solution: To determine the block diagram of the system, we first separate it into simpler components.

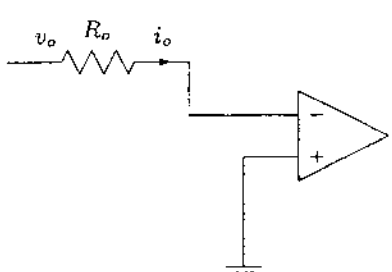


Because the input variable is v_i, we write i_i in terms v_i, such that

$$I_i(s) = \frac{1}{R_i}V_i(s),$$

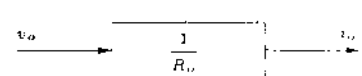
since the operational amplifier is assumed to be ideal.

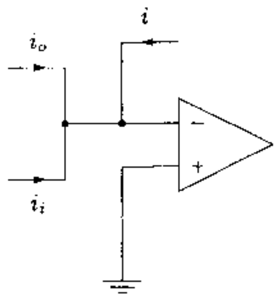




Similarly, we have

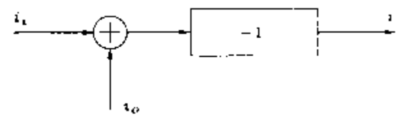
$$I_o(s) = \frac{1}{R_o}V_o(s).$$

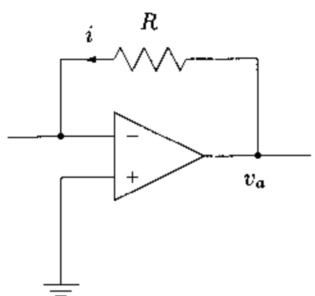




For an ideal operational amplifier,

$$i(t) = -\big(i_i(t) + i_o(t)\big).$$



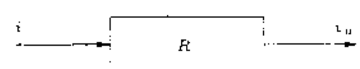


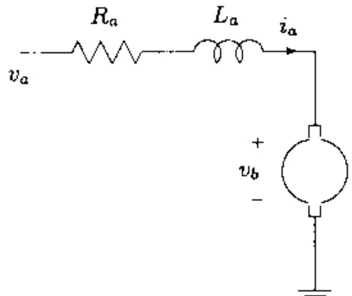
Again for an ideal operational amplifier,

$$v_a(t) = Ri(t),$$

or

$$V_a(s) = RI(s).$$



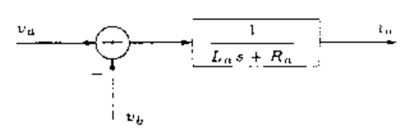


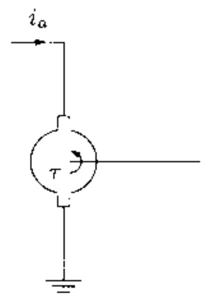
The armature current of the motor can be obtained from the Kirchhoff's Voltage Law, where

$$L_a \frac{\mathrm{d}i_a(t)}{\mathrm{d}t} + R_a i_a(t) + v_b(t) = v_a(t),$$

or

$$I_a(s) = \frac{1}{L_a s + R_a}\big(V_a(s) - V_b(s)\big).$$



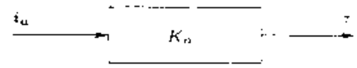


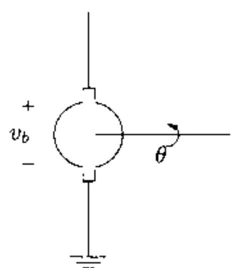
From the armature controlled motor,

$$\tau(t) = K_a i_a(t),$$

or

$$T(s) = K_a I_a(s).$$



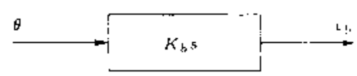


The back-emf voltage of the motor

$$v_b(t) = K_b \frac{\mathrm{d}\theta(t)}{\mathrm{d}t},$$

or

$$V_b(s) = (K_b s)\Theta(s).$$

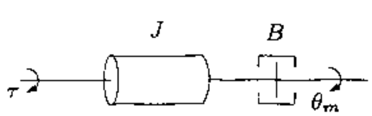


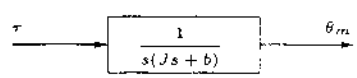
The torque equation is

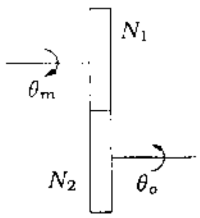
$$J\frac{\mathrm{d}^2\theta_m(t)}{\mathrm{d}t^2} = \tau(t) - B\frac{\mathrm{d}\theta(t)}{\mathrm{d}t},$$

or

$$\Theta_m(s) = \frac{1}{s(Js+B)}T(s).$$

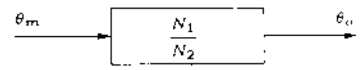






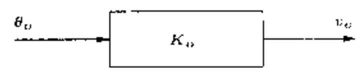
The gear relates the two angle values, such that

$$\theta_o(t) = \frac{N_1}{N_2}\theta_m(t).$$

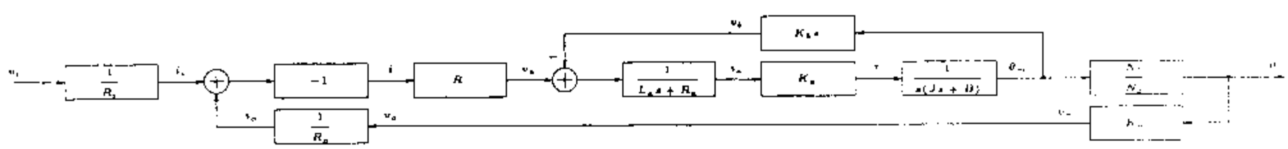


And, finally the given relationship

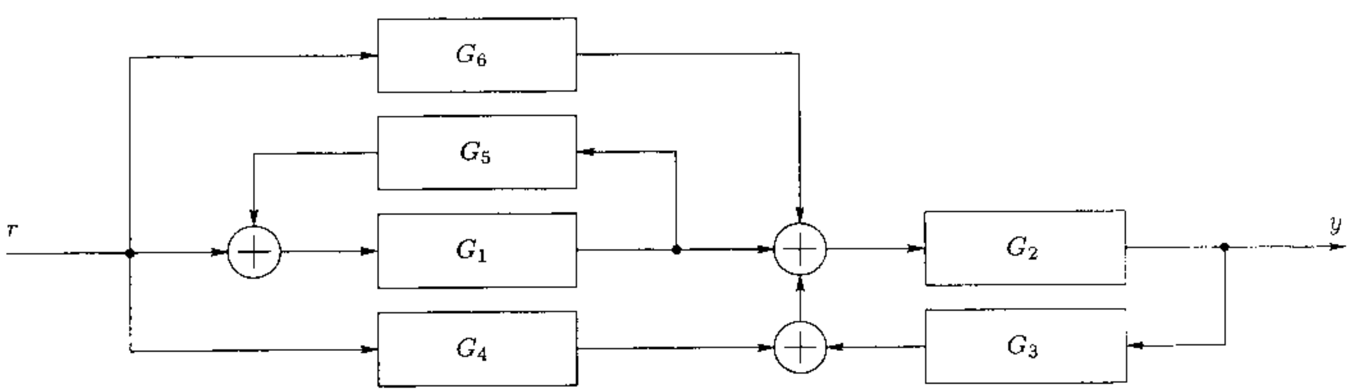
$$v_o(t) = K_o\theta_o(t).$$



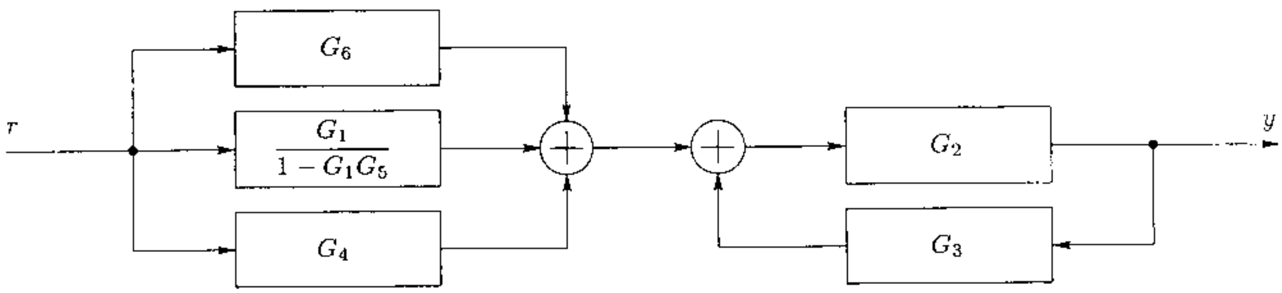
When we connect all the individual blocks together, we get the following block diagram.

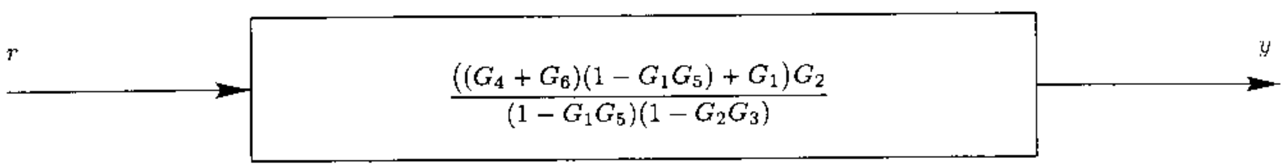


3. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly.

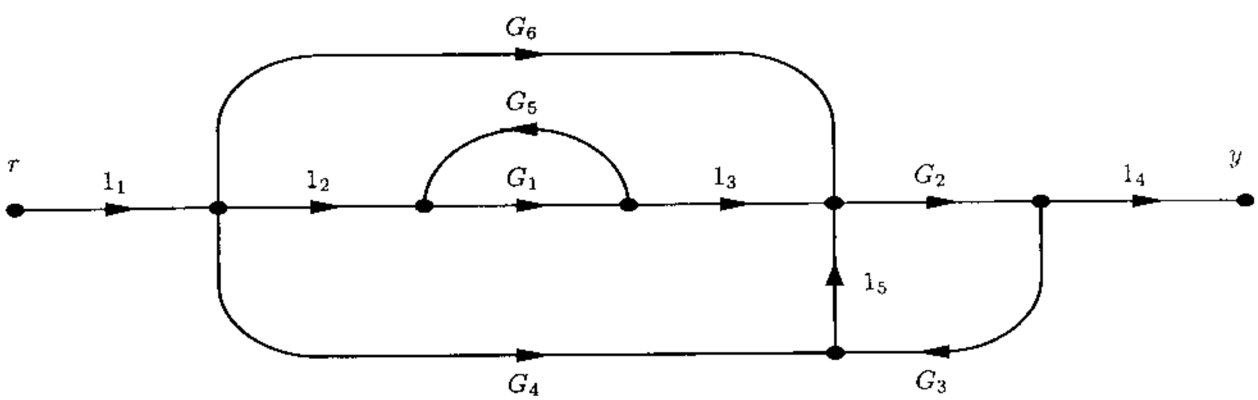


Solution: If we choose to use the block-diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.





If we choose to use Mason's formula, we need to draw the signal flow graph of the block diagram.



In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The forward path gains are

$$F_1 = 1_1 1_2 G_1 1_3 G_2 1_4 = G_1 G_2,$$

$$F_2 = 1_1 G_6 G_2 1_4 = G_2 G_6,$$

and

$$F_3 = 1_1 G_4 1_5 G_2 1_4 = G_2 G_4.$$

The loop gains are

$$L_1 = G_1 G_5,$$

and

$$L_2 = G_2 G_3 1_5 = G_2 G_3.$$

From the forward path and the loop gains, we determine the touching loops and the forward paths.

Touching Loops

	L_1	L_2
L_1	✔	✘
L_2		✔

Loops on Forward Paths

	L_1	L_2
F_1	✔	✔
F_2	✔	✘
F_3	✔	✘

Therefore,

$$\begin{aligned}\Delta &= 1 - (L_1 + L_2) + (L_1 L_2)\\ &= 1 - \big((G_1G_5) + (G_2G_3)\big) + \big((G_1G_5)(G_2G_3)\big)\\ &= 1 - G_1G_5 - G_2G_3 + G_1G_2G_3G_5,\end{aligned}$$

and

$$\begin{aligned}\Delta_1 &= \Delta\big|_{L_1=L_2=0} = 1,\\ \Delta_2 &= \Delta\big|_{L_2=0} = 1 - L_1 = 1 - G_1G_5,\\ \Delta_3 &= \Delta\big|_{L_2=0} = 1 - L_1 = 1 - G_1G_5.\end{aligned}$$

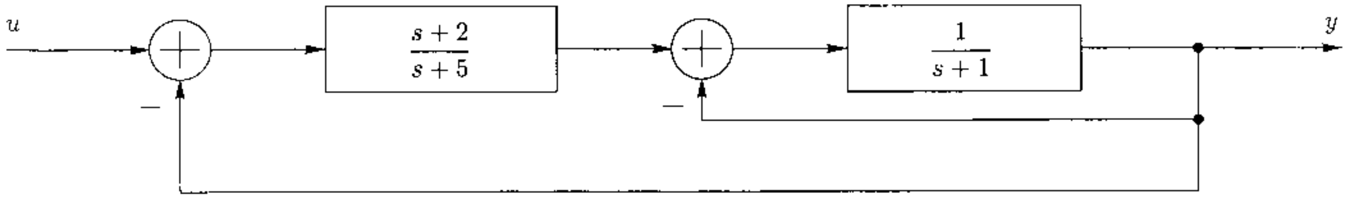
So,

$$\frac{Y(s)}{R(s)} = \frac{1}{\Delta}\sum_{i=1}^{3} F_i\Delta_i = \frac{(G_1G_2)(1) + (G_2G_6)(1 - G_1G_5) + (G_2G_4)(1 - G_1G_5)}{1 - G_1G_5 - G_2G_3 + G_1G_2G_3G_5},$$

or

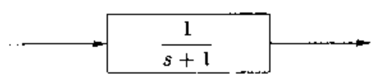
$$\frac{Y(s)}{R(s)} = \frac{G_1G_2 + G_2G_6 - G_1G_2G_5G_6 + G_2G_4 - G_1G_2G_4G_5}{1 - G_1G_5 - G_2G_3 + G_1G_2G_3G_5}.$$

4. The block diagram of a control system is given below.

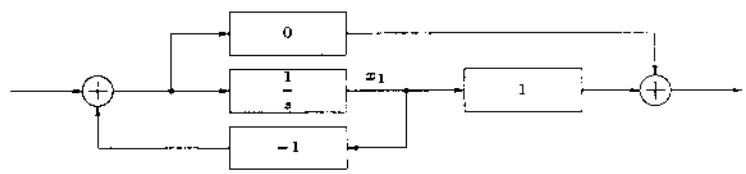


(a) Obtain a state-space representation of the system without any block-diagram reduction.

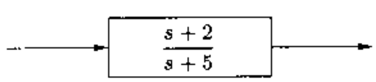
Solution: In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.



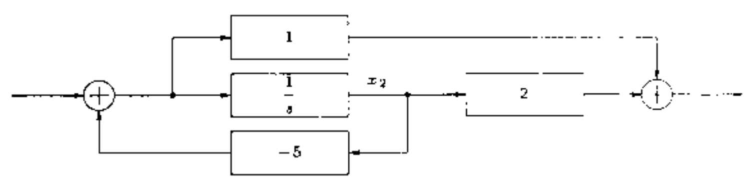
(a) The second feedforward gain block.



(b) Controller realization form.

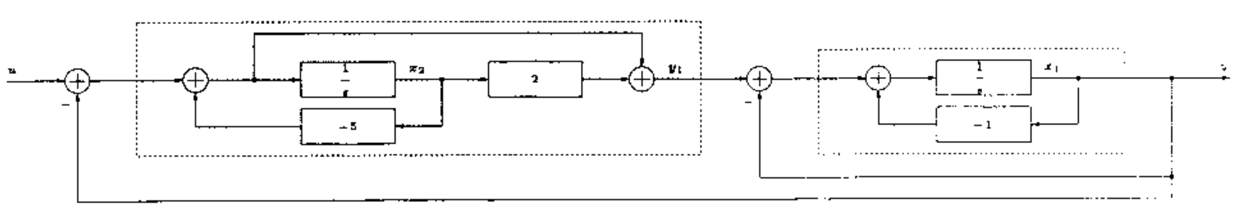


(c) The first feedforward gain block.



(d) Controller realization form.

The connected and "expanded" block diagram is shown below.



After assigning the state variables as shown in the figure, we obtain

$$\dot{x}_1 = -x_1 + (y_1 - y),$$
$$\dot{x}_2 = -5x_2 + (u - y),$$

and

$$y = x_1,$$

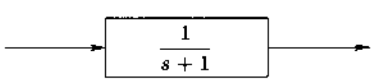
where

$$\begin{aligned} y_1 &= 2x_2 + \dot{x}_2 \\ &= 2x_2 + \left(-5x_2 + (u - y)\right) \\ &= -x_1 - 3x_2 + u. \end{aligned}$$

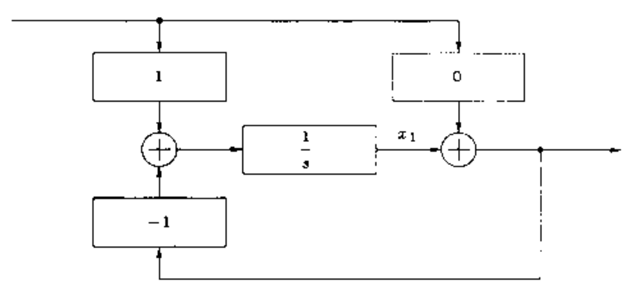
After eliminating the intermediate variable: y_1, we obtain the state-space representation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

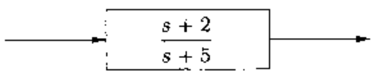
If we use the observer realization form for each of the blocks, then we obtain a different state-space representation.



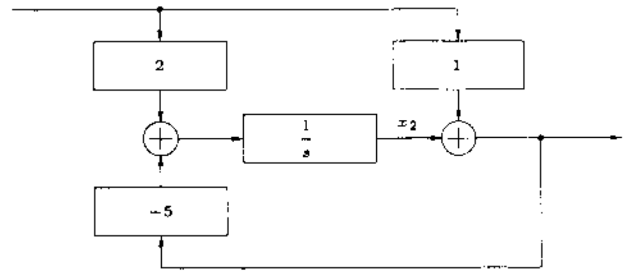
(a) The second feedforward gain block.



(b) Observer realization form.

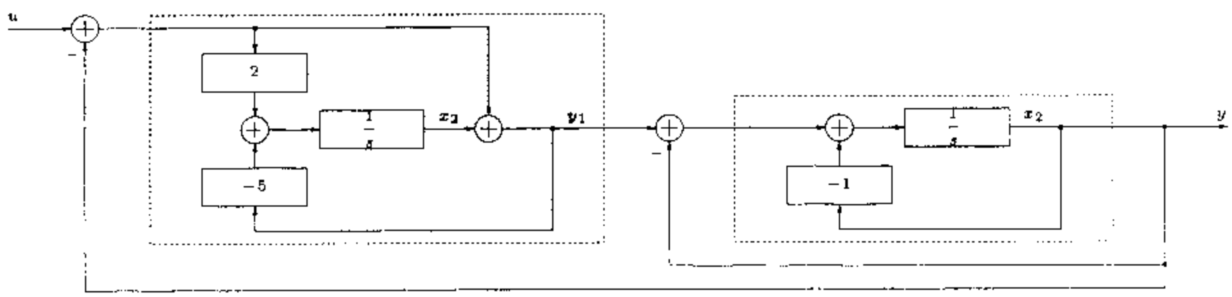


(c) The first feedforward gain block.



(d) Observer realization form.

The connected and "expanded" block diagram for this case is shown below.



Similarly, we obtain

$$\dot{x}_1 = -x_1 + (y_1 - y),$$
$$\dot{x}_2 = -5y_1 + 2(u - y),$$

and

$$y = x_1,$$

where

$$y_1 = x_2 + (u - y).$$

And,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

(b) Determine the transfer function of the system.

Solution: The transfer matrix of a system given in state-space representation is

$$F(s) = C(sI - A)^{-1}B + D.$$

For the controller realization form,

$$A = \begin{bmatrix} -3 & -3 \\ -1 & -5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \end{bmatrix},$$

and I is the appropriately dimensioned identity matrix. So,

$$\begin{aligned}
F(s) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & -3 \\ -1 & -5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s+3 & 3 \\ 1 & s+5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{(s+3)(s+5) - (1)(3)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+5 & -3 \\ -1 & s+3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{s^2 + 8s + 12} \begin{bmatrix} s+5 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{(s+2)(s+6)} \begin{bmatrix} s+2 \end{bmatrix} \\
&= \frac{s+2}{(s+2)(s+6)}.
\end{aligned}$$

Therefore, the transfer function is

$$F(s) = \frac{1}{s+6}.$$

For the observer realization form,

$$A = \begin{bmatrix} -3 & 1 \\ 3 & -5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ -3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \end{bmatrix},$$

and I is the appropriately dimensioned identity matrix. So,

$$F(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 3 & -5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s+3 & -1 \\ -3 & s+5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

[redacted]