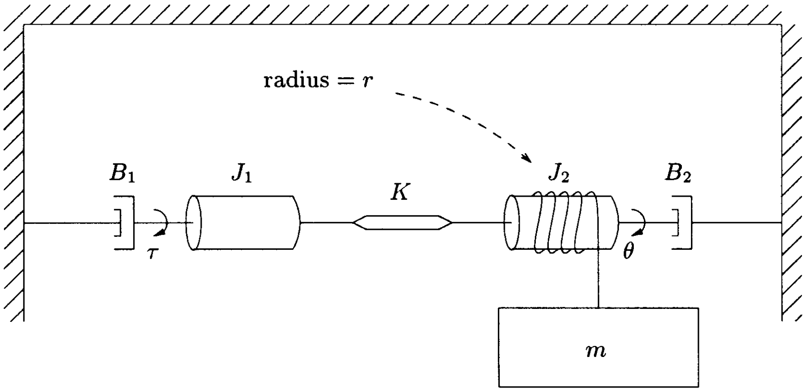
EE 231 | Exam#1 | Mar. 11, 2003

75 minutes

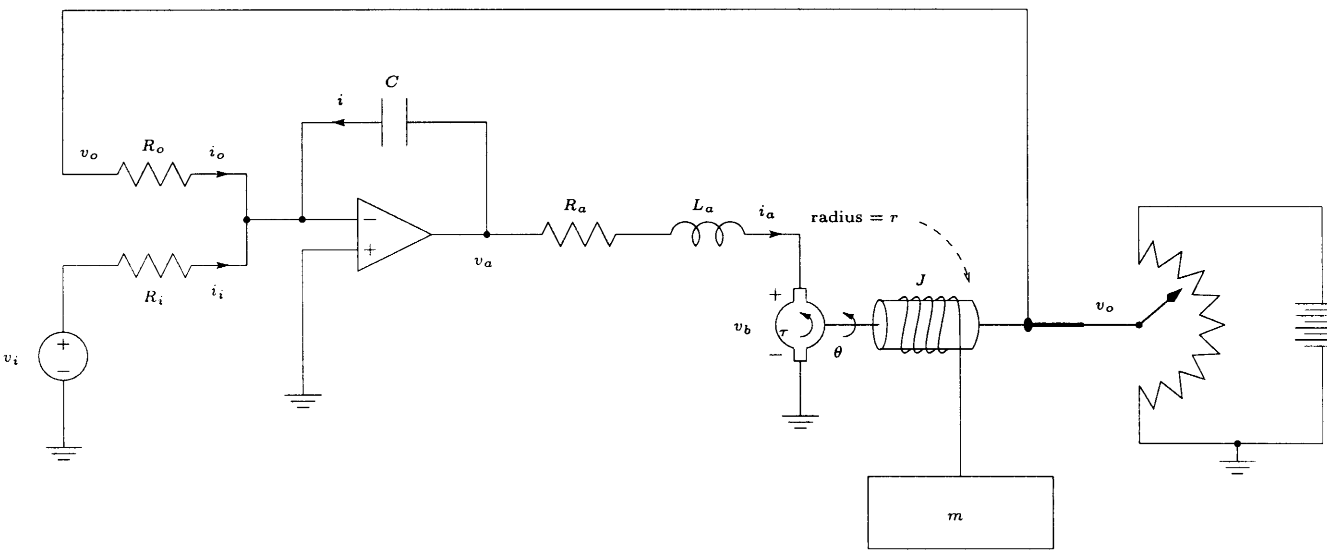
1. Consider the mechanical system shown below. Assume that g is the gravitational acceleration.



   (a) Obtain the transfer matrix of the system assuming that the external variables τ and g are the inputs, and the angle θ is the output. (15pts)

   (b) Obtain *either* the torque-voltage *or* the torque-current analog of the system. (10pts)

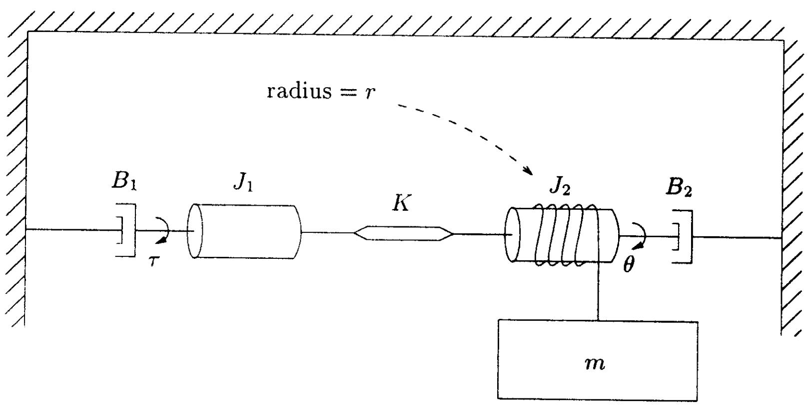
2. The angular position of the shaft of a motor is controlled by the system shown below.



The angular position of the motor shaft is detected by a variable resistor which provides a voltage v_o proportional to the angle, such that $v_o = K_o\theta$. Assume that g is the gravitational acceleration. Draw the most detailed block diagram of the system, where v_i is the input, and θ is the output. Show all the variables v_i, i_i, v_o, i_o, i, v_a, v_b, i_a, τ, and θ on the block diagram. (25pts)

EE 231 — Exam#1 Solutions — **Mar. 11, 2003**

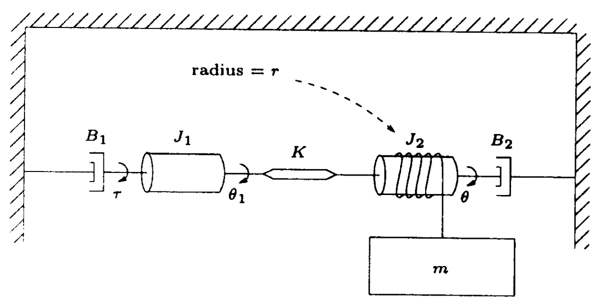
1. Consider the mechanical system shown below. Assume that g is the gravitational acceleration.



(a) Obtain the transfer matrix of the system assuming that the external variables τ and g are the inputs, and the angle θ is the output.

Solution:

First, we identify the linearly independent rotations in the mechanical system and mark them.



Then, we write the differential equations describing the motion from the mechanical system.

$$J_1\ddot{\theta}_1 = \tau - B_1\dot{\theta}_1 - K(\theta_1 - \theta)$$

$$J_2\ddot{\theta} = mgr - B_2\dot{\theta} - K(\theta - \theta_1),$$

since the attached mass generates a force of mg or a torque of mgr. Next, we obtain the transfer function by taking the Laplace transforms of the above equations under zero initial conditions. After some manipulations, we get

$$(J_1s^2 + B_1s + K)\Theta_1(s) - K\Theta(s) = T(s),$$

and

$$(J_2s^2 + B_2s + K)\Theta(s) - K\Theta_1(s) = mgr(1/s),$$

where Θ_1, Θ, and T are the Laplace transforms of θ_1, θ, and τ, respectively. After multiplying the first equation by K and substituting $K\Theta_1(s)$ from the second equation, we get

$$(J_1s^2 + B_1s + K)\big((J_2s^2 + B_2s + K)\Theta(s) - mgr(1/s)\big) - K^2\Theta(s) = KT(s),$$

or

$$\big((J_1s^2 + B_1s + K)(J_2s^2 + B_2s + K) - K^2\big)\Theta(s) = (J_1s^2 + B_1s + K)mgr(1/s) + KT(s).$$

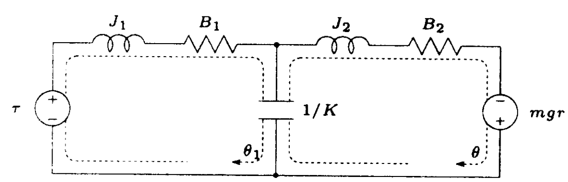
Therefore,

$$\Theta(s) = \frac{1}{(J_1s^2 + B_1s + K)(J_2s^2 + B_2s + K) - K^2}\begin{bmatrix} mr(J_1s^2 + B_1s + K)/s & K \end{bmatrix}\begin{bmatrix} g \\ T(s) \end{bmatrix}.$$

(b) Obtain *either* the torque-voltage *or* the torque-current analog of the system.

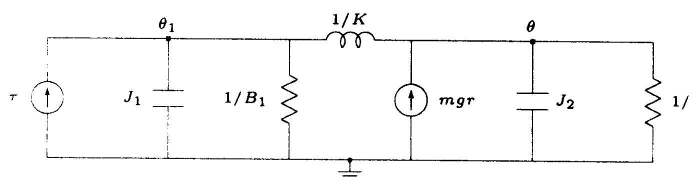
Solution: For the torque-voltage analog of a mechanical system, there will be a loop charge associated with each rotational variable, and an input torque will be associated with a voltage source. The stiffness constant, the rotational damping-constant, and the inertia will be associated with the reciprocal of capacitance, the resistance, and the inductance, respectively. The elements between two rotational variables of the mechanical system will be between the corresponding loop variables of the torque-voltage analog. The elements that are connected to fixed frames and the elements that are always measured with respect to a fixed frame, such as the inertia and the external torque, will be on the non-common portions of the loops.

The next figure shows the torque-voltage analog of the mechanical system, where the loops are identified with the angles.

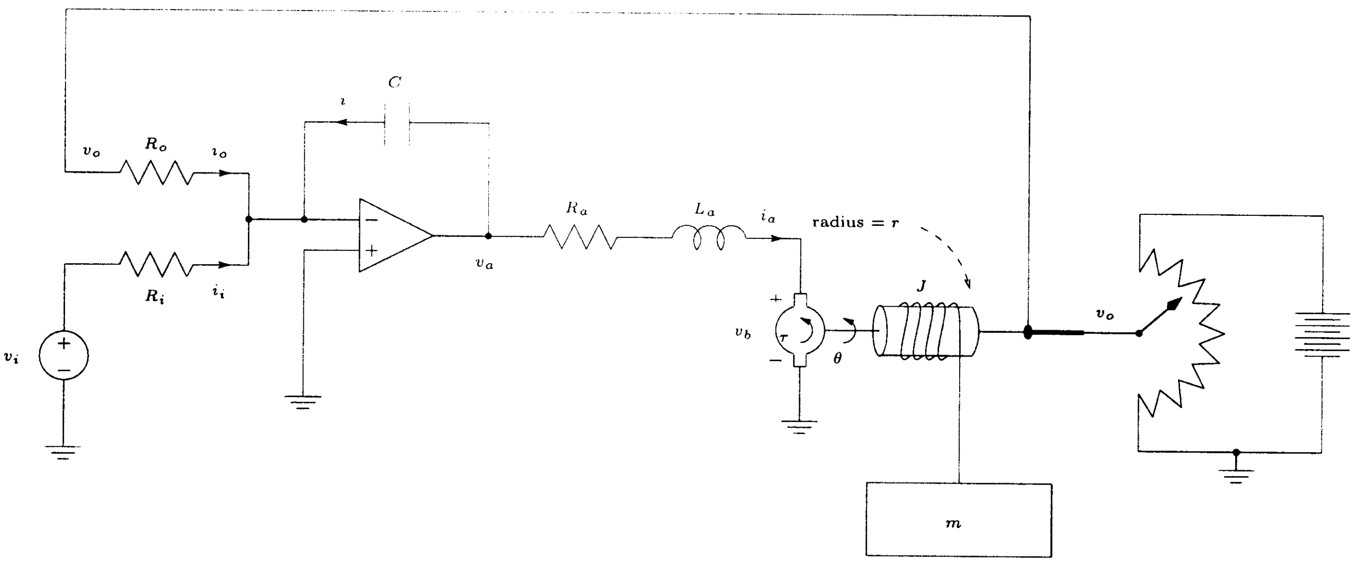


For the torque-current analog of a mechanical system, there will be a node flux associated with each rotational variable, and an input torque will be associated with a current source. The stiffness constant, the rotational damping-constant, and the inertia will be associated with the reciprocal of inductance, the conductance, and the capacitance, respectively. The elements between two rotational variables of the mechanical system will be between the corresponding node variables of the torque-current analog. The elements that are connected to fixed frames and the elements that are always measured with respect to a fixed frame, such as the inertia and the external torque, will be connected to the ground.

The next figure shows the torque-current analog of the mechanical system, where the nodes are identified with the angles.

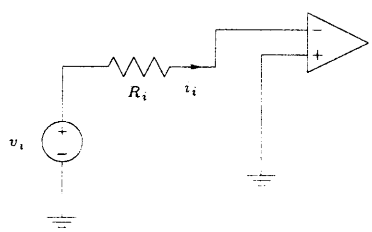


2. The angular position of the shaft of a motor is controlled by the system shown below.



The angular position of the motor shaft is detected by a variable resistor which provides a voltage v_o proportional to the angle, such that $v_o = K_o\theta$. Assume that g is the gravitational acceleration. Draw the most detailed block diagram of the system, where v_i is the input, and θ is the output. Show all the variables v_i, i_i, v_o, i_o, i, v_a, v_b, i_a, τ, and θ on the block diagram.

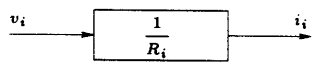
Solution: To determine the block diagram of the system, we first separate it into simpler components.

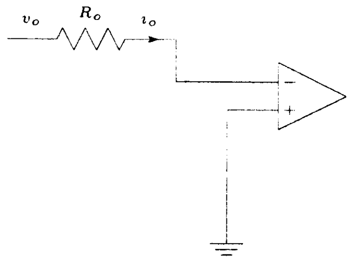


Since the input variable is v_i, we write i_i in terms v_i, such that

$$I_i(s) = \frac{1}{R_i} V_i(s),$$

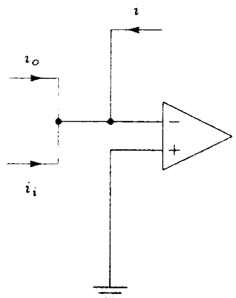
since the operational amplifier is assumed to be ideal.





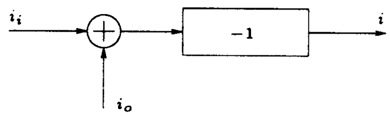
Similarly, we have

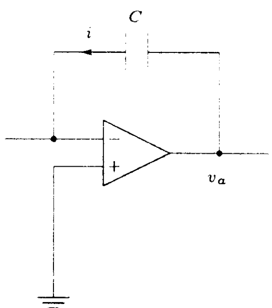
$$I_o(s) = \frac{1}{R_o} V_o(s).$$



For an ideal operational amplifier,

$$i(t) = -\big(i_i(t) + i_o(t)\big).$$



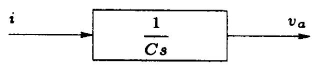


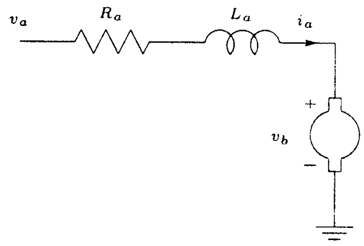
Again for an ideal operational amplifier,

$$v_a(t) = \frac{1}{C} \int^t i(\tau)\, \mathrm{d}\tau,$$

or

$$V_a(s) = \frac{1}{Cs} I(s).$$



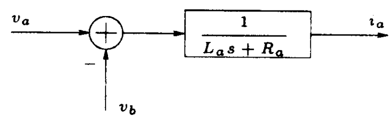


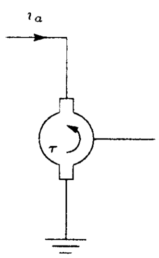
The armature current of the motor can be obtained from the Kirchhoff's Voltage Law, where

$$L_a \frac{\mathrm{d}i_a(t)}{\mathrm{d}t} + R_a i_a(t) + v_b(t) = v_a(t),$$

or

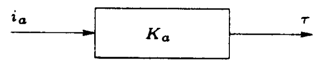
$$I_a(s) = \frac{1}{L_a s + R_a}\big(V_a(s) - V_b(s)\big).$$

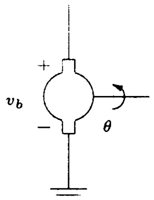




From the armature-controlled motor,

$$\tau(t) = K_a i_a(t).$$



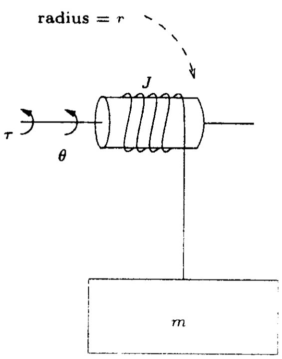


The back-emf voltage of the motor

$$v_b(t) = K_b \frac{\mathrm{d}\theta(t)}{\mathrm{d}t},$$

or

$$V_b(s) = (K_b s)\Theta(s).$$

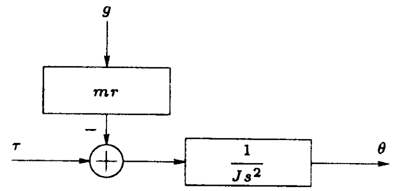


The torque equation for θ is

$$J\frac{\mathrm{d}^2\theta_m(t)}{\mathrm{d}t^2} = \tau(t) - mgr,$$

since the mass m generates a force of mg on the rope, and the corresponding torque is mgr. So,

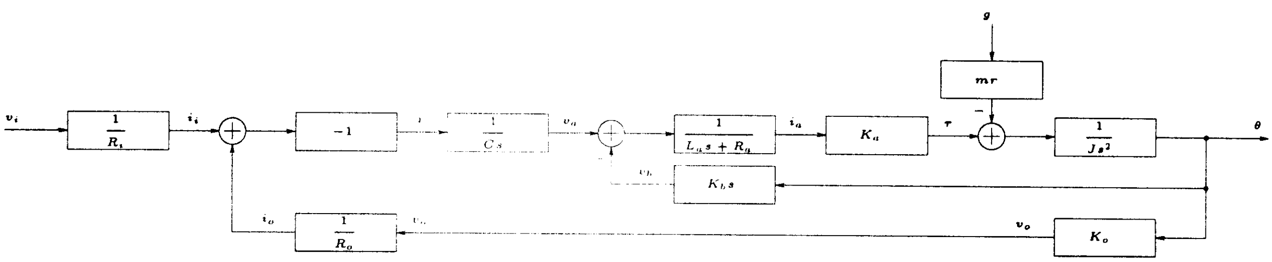
$$\Theta(s) = \frac{1}{Js^2}\big(T(s) - (mr)g\big).$$



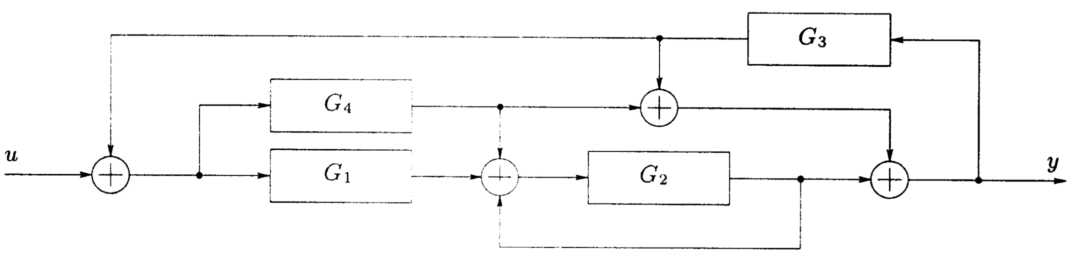
And, finally the given relationship

$$v_o(t) = K_o\theta(t).$$

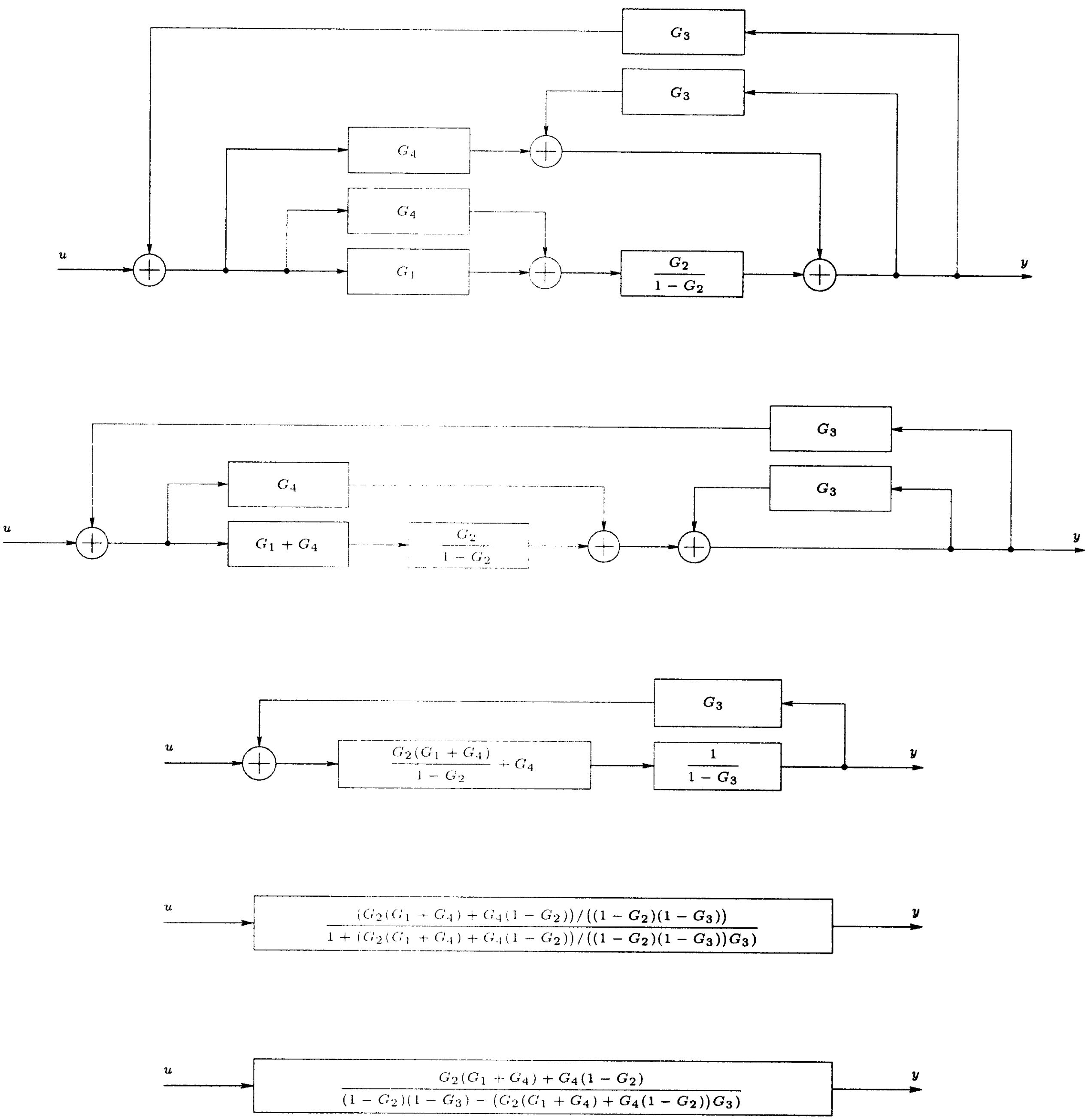
When we connect all the individual blocks together, we get the following block diagram.



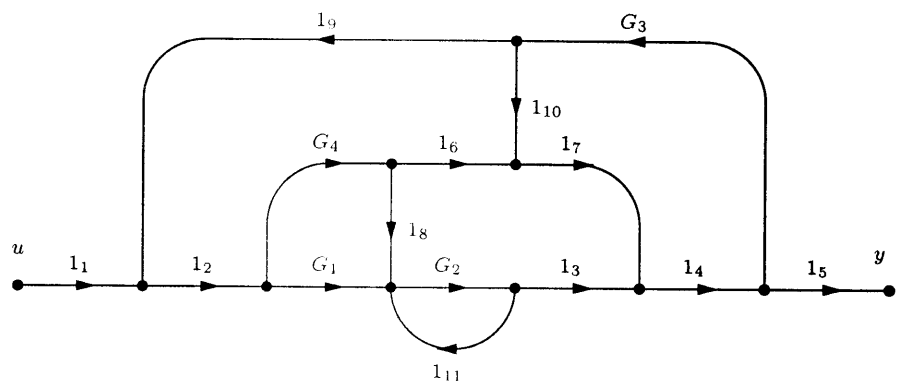
3. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly.



Solution: If we choose to use the block-diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.



If we choose to use Mason's formula, we need to draw the signal flow graph of the block diagram.



In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The forward path gains are

$$F_1 = 1_1 1_2 G_1 G_2 1_3 1_4 1_5 = G_1 G_2,$$

$$F_2 = 1_1 1_2 G_4 1_8 G_2 1_3 1_4 1_5 = G_2 G_4,$$

and

$$F_3 = 1_1 1_2 G_4 1_6 1_7 1_4 1_5 = G_4.$$

The loop gains are

$$L_1 = G_2 1_{11} = G_2,$$

$$L_2 = 1_2 G_1 G_2 1_3 1_4 G_3 1_9 = G_1 G_2 G_3,$$

$$L_3 = 1_2 G_4 1_8 G_2 1_3 1_4 G_3 1_9 = G_2 G_3 G_4,$$

$$L_4 = 1_2 G_4 1_6 1_7 1_4 G_3 1_9 = G_3 G_4,$$

and

$$L_5 = 1_7 1_4 G_3 1_{10} = G_3.$$

From the forward path and the loop gains, we determine the touching loops and the forward paths.

Touching Loops

	L_1	L_2	L_3	L_4	L_5
L_1	✔	✔	✔	✘	✘
L_2		✔	✔	✔	✔
L_3			✔	✔	✔
L_4				✔	✔
L_5					✔

Loops on Forward Paths

	L_1	L_2	L_3	L_4	L_5
F_1	✔	✔	✔	✔	✔
F_2	✔	✔	✔	✔	✔
F_3	✘	✔	✔	✔	✔

Therefore,

$$\begin{aligned}\Delta &= 1 - (L_1 + L_2 + L_3 + L_4 + L_5) + (L_1L_4 + L_1L_5)\\ &= 1 - ((G_2) + (G_1G_2G_3) + (G_2G_3G_4) + (G_3G_4) + (G_3)) + ((G_2)(G_3) + (G_2)(G_3G_4))\\ &= 1 - G_2 - G_1G_2G_4 - G_3G_4 - G_3 + G_2G_3,\end{aligned}$$

and

$$\begin{aligned}\Delta_1 &= \Delta|_{L_1=L_2=L_3=L_4=L_5=0} = 1,\\ \Delta_2 &= \Delta|_{L_1=L_2=L_3=L_4=L_5=0} = 1,\\ \Delta_3 &= \Delta|_{L_2=L_3=L_4=L_5=0} = 1 - L_1 = 1 - G_2.\end{aligned}$$

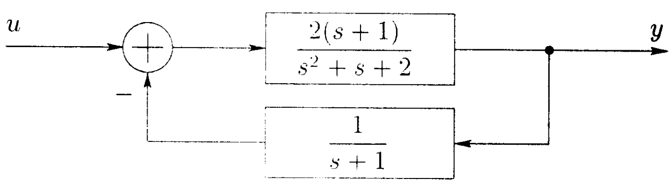
So,

$$\frac{Y(s)}{U(s)} = \frac{1}{\Delta}\sum_{i=1}^{3} F_i\Delta_i = \frac{(G_1G_2)(1) + (G_2G_4)(1) + (G_4)(1 - G_2)}{1 - G_2 - G_1G_2G_4 - G_3G_4 - G_3 + G_2G_3},$$

or

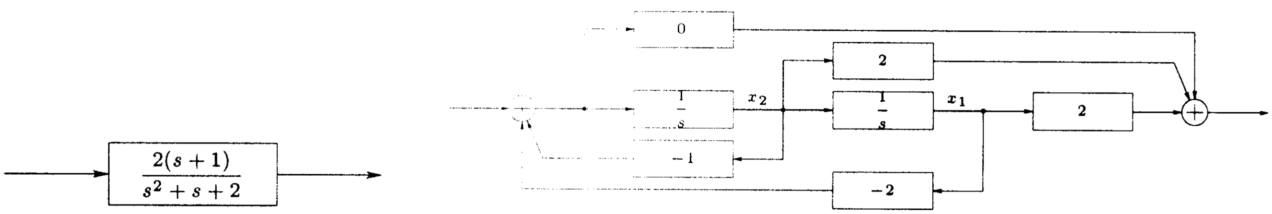
$$\frac{Y(s)}{U(s)} = \frac{G_1G_2 + G_4}{1 - G_2 - G_1G_2G_4 - G_3G_4 - G_3 + G_2G_3}.$$

4. The block diagram of a control system is given below.



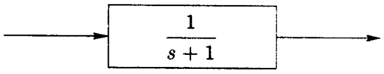
Obtain a state-space representation of the system without any block-diagram reduction.

Solution: In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.

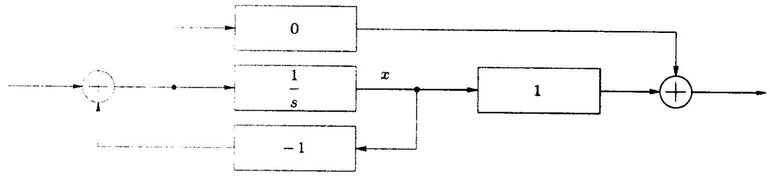


(a) The feedforward gain block.

(b) Controller realization form.

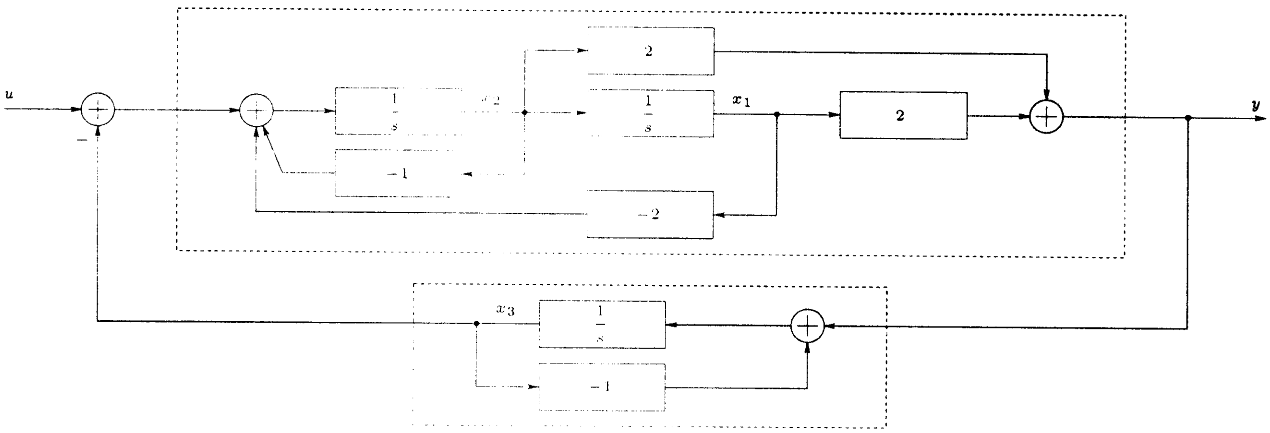


(c) The feedback gain block.



(d) Controller realization form.

The connected and "expanded" block diagram is shown below.



After assigning the state variables as shown in the figure, we obtain

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -2x_1 - x_2 + (u - x_3),$$

$$\dot{x}_3 = -x_3 + y,$$

and

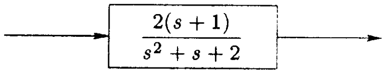
$$y = 2x_1 + 2x_2.$$

After eliminating the y variable from the last state-variable equation, we obtain the state-space representation

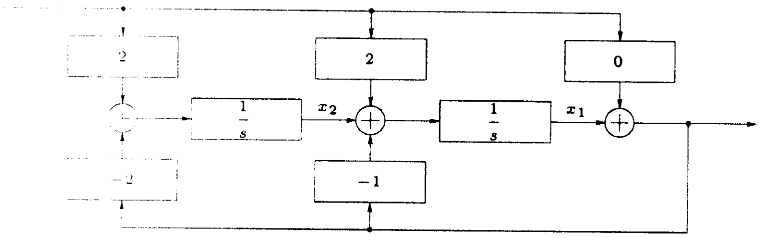
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -1 & -1 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

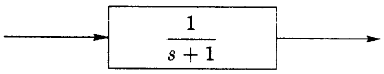
If we use the observer realization form for each of the blocks, then we obtain a different state-space representation.



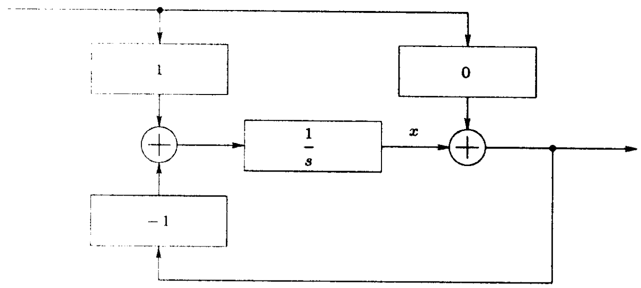
(a) The feedforward gain block.



(b) Observer realization form.

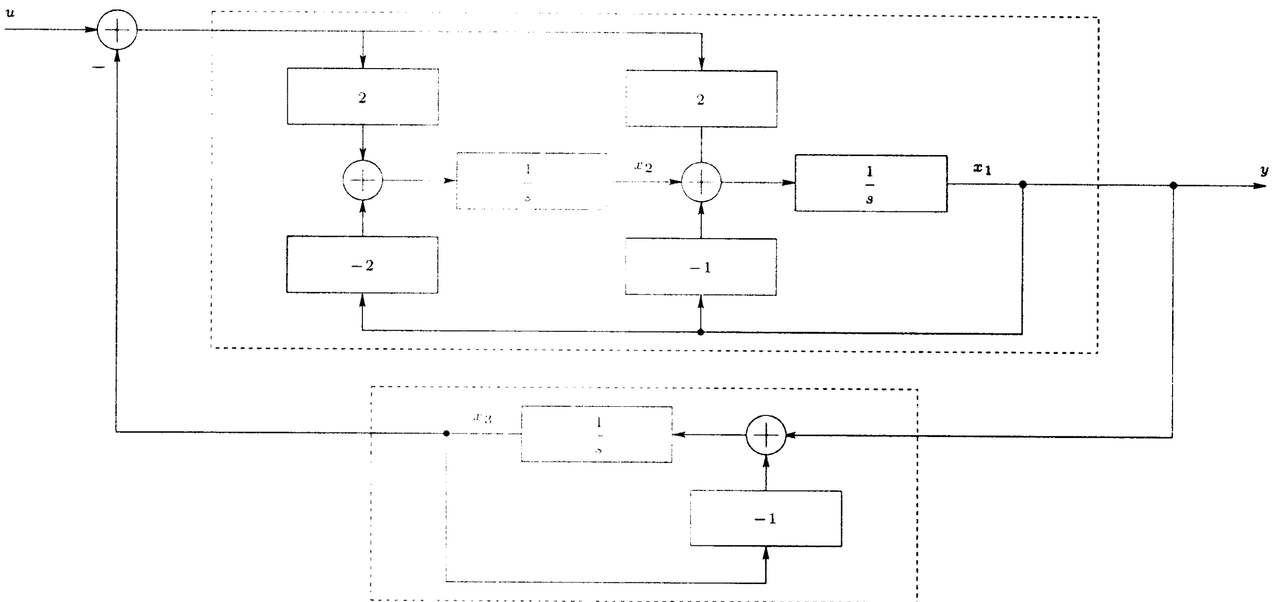


(c) The feedback gain block.



(d) Observer realization form.

The connected and "expanded" block diagram for this case is shown below.



Similarly, we obtain

$$\dot{x}_1 = -x_1 + x_2 + 2(u - x_3),$$
$$\dot{x}_2 = -2x_1 + 2(u - x_3),$$
$$\dot{x}_3 = -x_3 + y.$$

and

$$y = x_1.$$

And,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & -2 \\ -2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

5. A control system is described in state-space representation, such that

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -2 & 0 \\ -1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(t),$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Determine the transfer function or the transfer matrix of the system.

Solution: The transfer matrix of a control system described in the state-state representation

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t),$$

is

$$F(s) = C(sI - A)^{-1}B + D,$$

where

$$A = \begin{bmatrix} -2 & 0 \\ -1 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad D = 0,$$

and I is the appropriately dimensioned identity matrix. So,

$$F(s) = \begin{bmatrix} 1 & 1 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ -1 & -2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 1 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

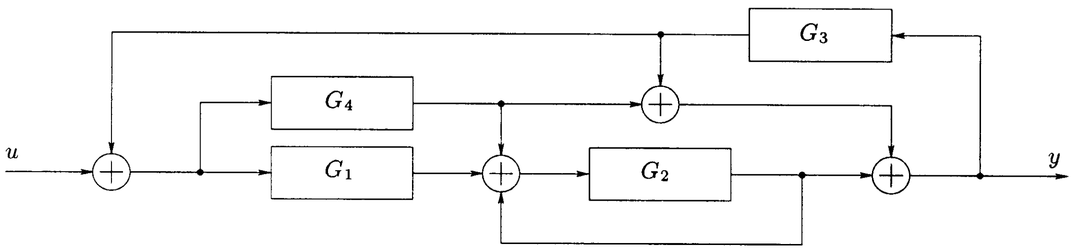
$$= \frac{1}{(s+2)^2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ -1 & s+2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s+2)^2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ s+2 \end{bmatrix}$$

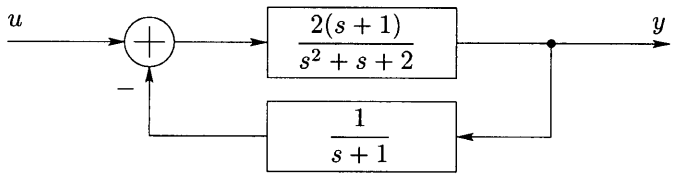
$$= \frac{1}{(s+2)^2} (s+2).$$

In other words, the transfer function is $F(s) = 1/(s+2)$.

3. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly. (25pts)



4. The block diagram of a control system is given below.



Obtain a state-space representation of the system without any block-diagram reduction. (15pts)

5. A control system is described in state-space representation, such that

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -2 & 0 \\ -1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(t),$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Determine the transfer function or the transfer matrix of the system. (10pts)