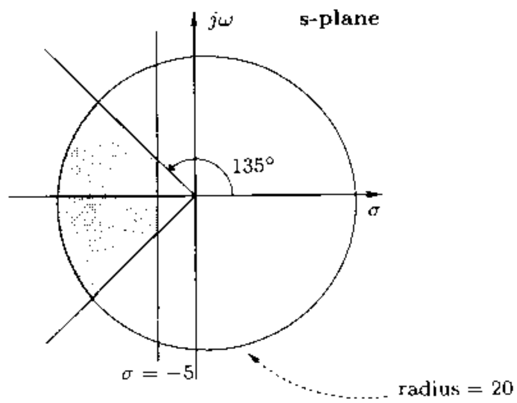
EE 231 | **Exam#2** | May 02, 2002

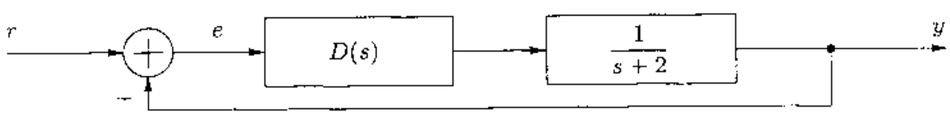
75 minutes

1. (a) Obtain the necessary inequalities to describe the strictly complex poles in the shaded region below in terms of only ζ and ω_n of a second-order system described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$. (10pts)



   (b) Consider a second-order system with no zero, such that its poles are located in the shaded region above. Determine the largest possible maximum percent-overshoot and the largest possible 2% settling-time of the system. (15pts)

2. For the following feedback control system, design the simplest controller $D(s)$ that would result in a zero steady-state error $e(\infty)$ for a ramp input. (20pts)



3. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{s^2 + 2s + 101}{(s+1)(s^2+2s+2)} = K\frac{s^2+2s+101}{s^3+3s^2+4s+2}.$$

   (a) Determine the values of K such that the closed-loop system is asymptotically stable. (20pts)

   (b) Determine the value (or values) of K and the natural frequency (or frequencies), such that the closed-loop system would have sustained oscillations. (10pts)

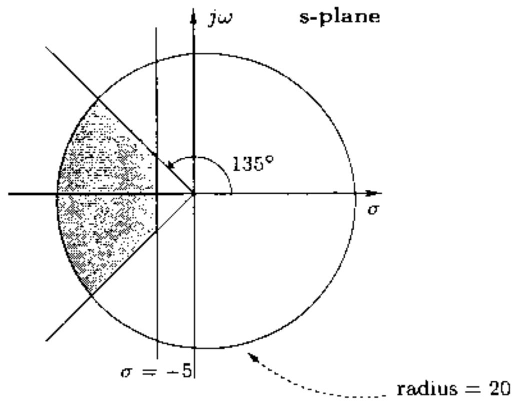
4. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{s+3}{(s+1)(s^2+2s+2)} = K\frac{s+3}{s^3+3s^2+4s+2}.$$

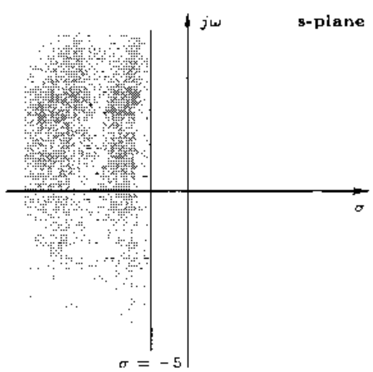
   Construct the root-locus diagram. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angle of arrivals and/or departures. (25pts)

Solutions

1. (a) Obtain the necessary inequalities to describe the strictly complex poles in the shaded region below in terms of only ζ and ω_n of a second-order system described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$.



Solution: To be able to describe the shaded region, we need to separate it into unions or intersections of simpler regions.

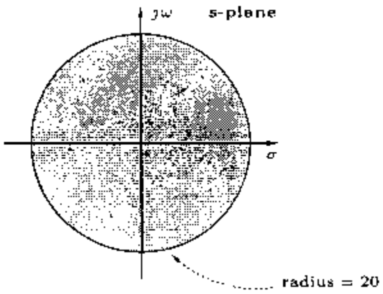


A vertical straight line designates a constant value for the real part of the poles. Since the real part of the complex poles are at $-\zeta\omega_n$, the shown shaded area is represented by

$$-\zeta\omega_n \leq -5,$$

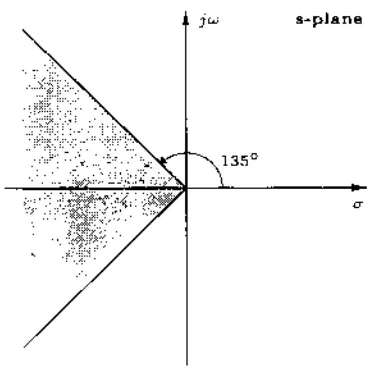
or

$$\zeta\omega_n \geq 5.$$



The equi-distance points from the origin designate constant value for ω_n. As a result, the shown shaded area is represented by

$$\omega_n \leq 20.$$



A straight line originating from the origin designates a constant ζ value, where $\cos^{-1}(\zeta)$ is the acute angle between the line and the negative real axis. So for the shaded area shown, we have

$$0° < \cos^{-1}(\zeta) \leq 180° - 135°,$$

or

$$\cos(0°) > \zeta \geq \cos(45°),$$

since $\cos(\theta)$ is a monotonically decreasing function for $0 < \theta < 180°$. So, we have

$$\frac{\sqrt{2}}{2} \leq \zeta < 1,$$

when the poles have non-zero imaginary parts.

Therefore, the shaded area given in the problem is the intersection of the individual shaded areas, and it can be represented by

$$\sqrt{1-\zeta^2}\omega_n \leq 10,$$

$$\omega_n \leq 20,$$

$$\sqrt{2}/2 \leq \zeta < 1.$$

(b) Consider a second-order system with no zero, such that its poles are located in the shaded region above. Determine the largest possible maximum percent-overshoot and the largest possible 2% settling-time of the system.

Solution: Maximum overshoot for a second-order system with no zero is given by

$$M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}.$$

The only system parameter that affects the maximum overshoot is ζ. For maximum M_p, we need to have minimum ζ; since $\zeta = 0$ gives undamped oscillations. In the shaded region, the minimum $\zeta = \sqrt{2}/2$, and the corresponding maximum overshoot is

$$M_p = e^{-\frac{\sqrt{2}/2}{\sqrt{1-(\sqrt{2}/2)^2}}\pi} = e^{-\pi} \approx 0.0432,$$

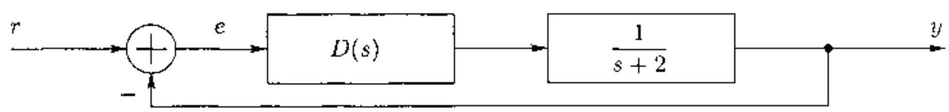
or the largest possible maximum percent-overshoot is 4.32%.

The 2% settling time of a second-order system with no zero is given by

$$t_{2\%s} = \frac{4}{\sigma_o} = \frac{4}{\zeta\omega_n}.$$

The only system parameter that affects the settling time is σ_o. For maximum $t_{2\%s}$, we need to have minimum σ_o. In the shaded region, the minimum $\sigma_o = 5$, and as a result the largest possible 2% settling time is 4/5 s.

2. For the following feedback control system, design the simplest controller $D(s)$ that would result in a zero steady-state error $e(\infty)$ for a ramp input.

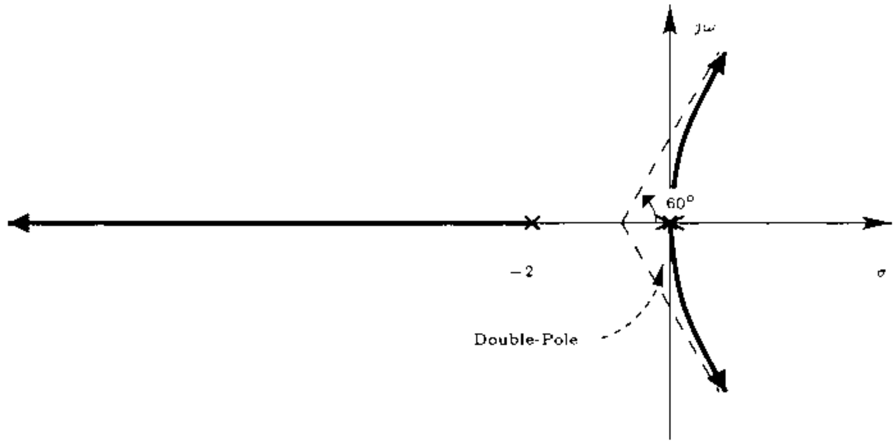


Solution: In order to have a zero steady-state error for any given input, we need to match the non-asyptotically stable poles of the input in the open-loop gain of the system. In the case of the ramp input, we need to have two poles as $s = 0$, or the system has to be of type-2. In other words,

$$D(s) = \frac{1}{s^2}D'(s),$$

for some $D'(s)$. Since there is no other explicit requirement, we only need to ensure stability by a proper and simple choice of $D'(s)$.

The simplest choice is $D'(s) = K$ for a constant K. We may use a number of methods to check the stability of the system for this choice, but a rough sketch of the root-locus, as shown below, is simple enough to see the location of the closed-loop poles.



As we observe from the root-locus diagram, there is no value of K that would result in a stable closed-loop system. As a consequence of the three poles and no zero, the asymptote angles are $\pm 60^\circ$ and 180°. In order to change the asymptote angles, we need to include zeros. Inclusion of a single zero will result in $\pm 90^\circ$ asymptote angles. As long as the real-axis crossing of the asymptotes is in the left-half plane, there will be a stable set of closed-loop poles. The real-axis crossing of the asymptotes is given by

$$\sigma_a = \frac{\sum p_i - \sum z_i}{n - m},$$

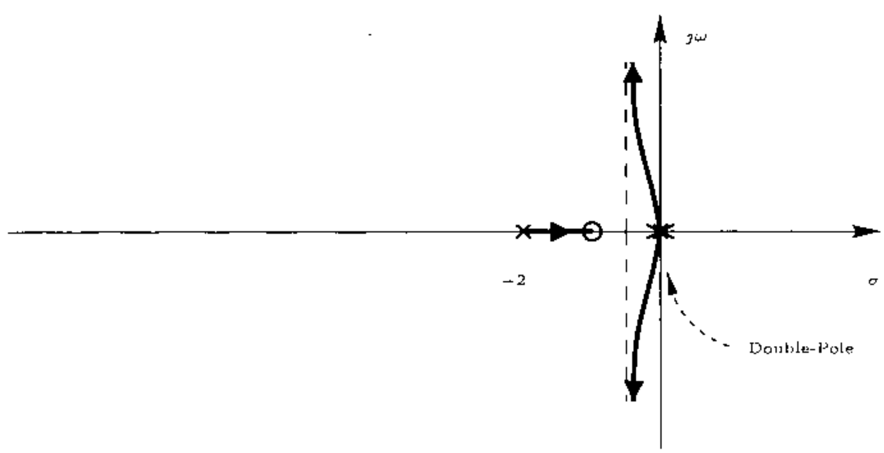
where $\sum p_i$ and $\sum z_i$ are the sums of the pole and zero locations, respectively. In our case, when

$$D(s) = K\frac{(s-z)}{s^2},$$

the real-axis crossing of the asymptote will be at

$$\sigma_a = \frac{\big((-2)+(0)+(0)\big) - \big((z)\big)}{3-1} = -\frac{z_1+2}{2}.$$

For $\sigma_a < 0$, we need to have $z > -2$. Obviously, the zero should also be in the left-half plane, since a right-half plane zero would generate a root-locus branch on the positive real axis. A sketch of the root-locus diagram for a zero between -2 and 0 is given in the figure below.



As we observe from the root-locus diagram, any positive value of K will generate asymptotically stable closed-loop poles. Therefore, the simplest controller is

$$D(s) = K\frac{(s-z)}{s^2},$$

where $-2 < z < 0$, and $K > 0$.

3. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{s^2+2s+101}{(s+1)(s^2+2s+2)} = K\frac{s^2+2s+101}{s^3+3s^2+4s+2}.$$

(a) Determine the values of K such that the closed-loop system is asymptotically stable.

Solution: The stability of the closed-loop system can be determined using the Routh-Hurwitz's stability criterion on the characteristic polynomial. From the characteristic equation, $1+G(s)=0$, we have

$$1 + K\frac{s^2+2s+101}{s^3+3s^2+4s+2} = 0,$$

or

$$s^3 + (K+3)s^2 + (2K+4)s + (101K+2) = 0.$$

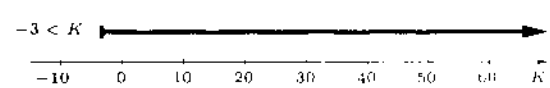
The Routh-Hurwitz table for the system becomes as given below.

s^3	1	$2K+4$
s^2	$K+3$	$101K+2$
s	$-\frac{(1)(101K+2)-(K+3)(2K+4)}{K+3} = \frac{2K^2-91K+10}{K+3}$	
1	$101K+2$	

The Routh-Hurwitz's stability criterion implies the following conditions.

i. $K+3>0$.

$$K > -3.$$



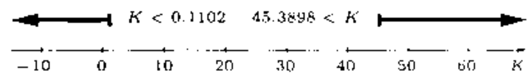
ii. $\frac{2K^2-91K+10}{K+3} > 0$.

A. $K + 3 > 0$ Case:

$$2K^2 - 91K + 10 > 0.$$
$$2(K - 0.1102)(K - 45.3898) > 0,$$

or

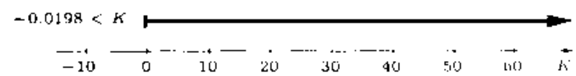
$$K < 0.1102, \text{ or } 45.3898 < K.$$



B. $10 + K < 0$ Case:
This case results in instability from the previous condition.

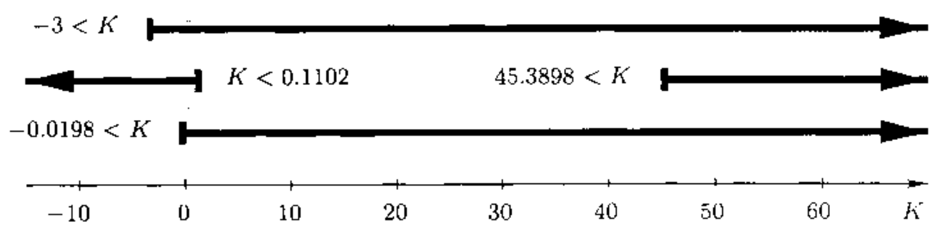
iii. $101K + 2 > 0$.

$$101K > -2.$$
$$K > -\frac{2}{101} = -0.0198.$$



The intersection of all these regions leads to

$$-0.0198 < K < 0.1102, \text{ or } 45.3898 < K.$$



(b) Determine the value (or values) of K and the natural frequency (or frequencies), such that the closed-loop system would have sustained oscillations.

Solution: For sustained oscillations, we need to choose K, such that there are distinct poles on the imaginary axis and no pole on the right-half plane. The candidates for such a choice are obtained by generating a row of zeros on the Routh-Hurwitz table. Observing from the table. the only two such rows are the s and the 1 rows. However, the 1 row gives an imaginary-axis crossing at $s = 0$. Considering the elements on the s row, we get

$$\frac{2K^2 - 91K + 10}{K + 3} = 0,$$

or

$$2K^2 - 91K + 10 = 0.$$

The solution of the above equation gives $K = 0.1102$ and $K = 45.3898$.

Next, we need to obtain the factors of the original polynomial from the previous row, and verify that we get poles on the imaginary axis. From the upper or the s^2 row,

$$\left((K+3)s^2 + (101K+2)\right)_{K=0.1102,45.3898} = 0.$$

Note here that the above equation gives some of the poles of the closed-loop system *only* for the values of K that make the s row all zero.

For $K = 0.1102$, we get $s = \pm j2.0543$, and for $K = 45.3898$, we get $s = \pm j9.7355$. So for both of the cases, we have imaginary-axis crossings. And, from the first elements of the remaining rows of the Routh-Hurwitz table, we conclude that the rest of the poles are in the left-half plane. Therefore, the natural frequencies, such that the closed-loop system would have sustained oscillations, are $\omega_1 = 2.0543\,\text{rad/s}$ and $\omega_2 = 9.7355\,\text{rad/s}$.

4. Consider a negative unity-feedback control system with the open-loop transfer function

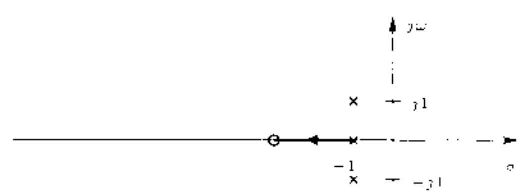
$$G(s) = K\frac{s+3}{(s+1)(s^2+2s+2)} = K\frac{s+3}{s^3+3s^2+4s+2}.$$

Construct the root-locus diagram. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angle of arrivals and/or departures.

Solution: First, we sketch the pole-zero locations and the real-axis portion of the root-locus diagram. Then, we decide the important features to be determined.

Need to determine:

- Asymptotes, and
- Angle of departures.



If the aymptotes are in the right-half plane, we may also need to determine imaginaty-axis crossings. However, there is no need to determine break-away or break-in points, since there is none.

Asymptotes

Real-Axis Crossing: $\sigma_a = \dfrac{\sum p_i - \sum z_i}{n-m}$

The real-axis crossing of the asymptotes is at

$$\sigma_a = \frac{\sum_i p_i - \sum_i z_i}{n-m} = \frac{\left((-1)+(-1+j1)+(-1-j1)\right) - \left((-3)\right)}{3-1} = 0.$$

Real-Axis Angles: $\theta_a = \dfrac{\pm(2k+1)\pi}{n-m}$

The angles that the asymptotes make with the real axis are determined from

$$\theta_a = \frac{\pm(2k+1)\pi}{n-m} = \frac{\pm(2k+1)\pi}{3-1} = \pm\frac{\pi}{2}.$$

Angle of Departure: $\sum \measuredangle(\cdot) = \pm(2k+1)\pi$

The angles of departures from complex open-loop poles are determined from the angular conditions about the open-loop poles. Therefore, the angular condition about $s = -1 + j1$ is

$$\measuredangle\big(s-(-3)\big) - \measuredangle\big(s-(-1)\big) - \measuredangle\big(s-(-1-j1)\big) - \measuredangle\big(s-(-1+j1)\big) = 180^\circ + k360^\circ.$$

$$\tan^{-1}\left(\frac{(1)-(0)}{(-1)-(-3)}\right) - \tan^{-1}\left(\frac{(1)-(0)}{(-1)-(-1)}\right) - \tan^{-1}\left(\frac{(1)-(-1)}{(-1)-(-1)}\right) - \theta_{\text{dep}} = 180^\circ + k360^\circ.$$

or

$$26.57^\circ - 90^\circ - 90^\circ - \theta_{\text{dep}} = 180^\circ + k360^\circ.$$

As a result,

$$\theta_{\text{dep}} = 26.57^\circ.$$

With the features determined, we can now sketch the root-locus diagram.