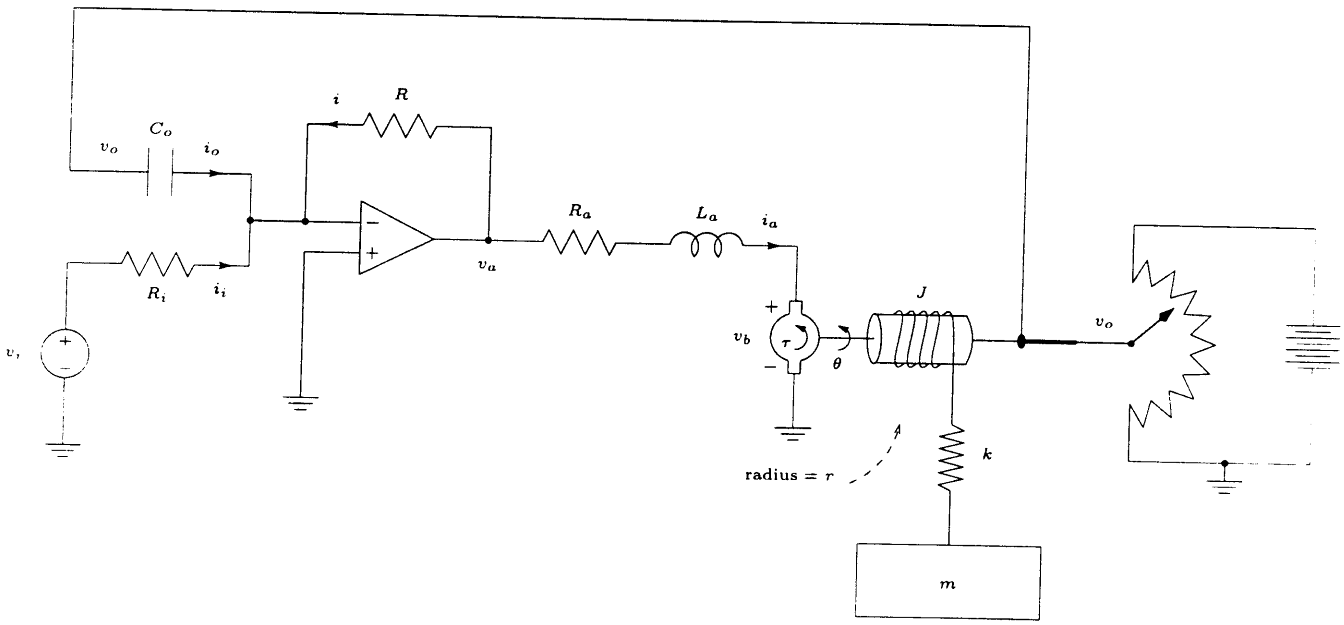
EE 231

Exam#1
75 minutes

Oct. 18, 2005

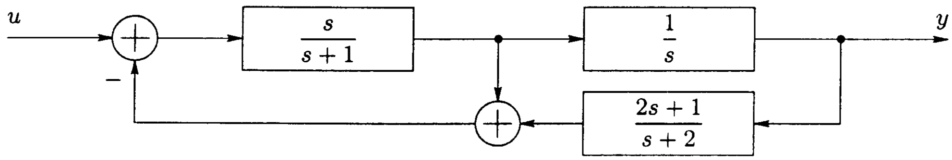
1. The angular position of the shaft of a motor is controlled by the system shown below.



The angular position of the motor shaft is detected by a variable resistor which provides a voltage v_o proportional to the angle, such that $v_o = K_o\theta$. Draw the most detailed block diagram of the system, where v_i is the input, and θ is the output. Show all the variables v_i, i_i, v_o, i_o, v_a, i, i_a, v_b, τ, and θ as well as the displacement(s) associated with the mass-spring components on the block diagram. (25pts)

2. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly. (25pts)

3. The block diagram of a control system is given below.



Obtain a state-space representation of the system without any block-diagram reduction. (25pts)

4. The state equation of a control system is given by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

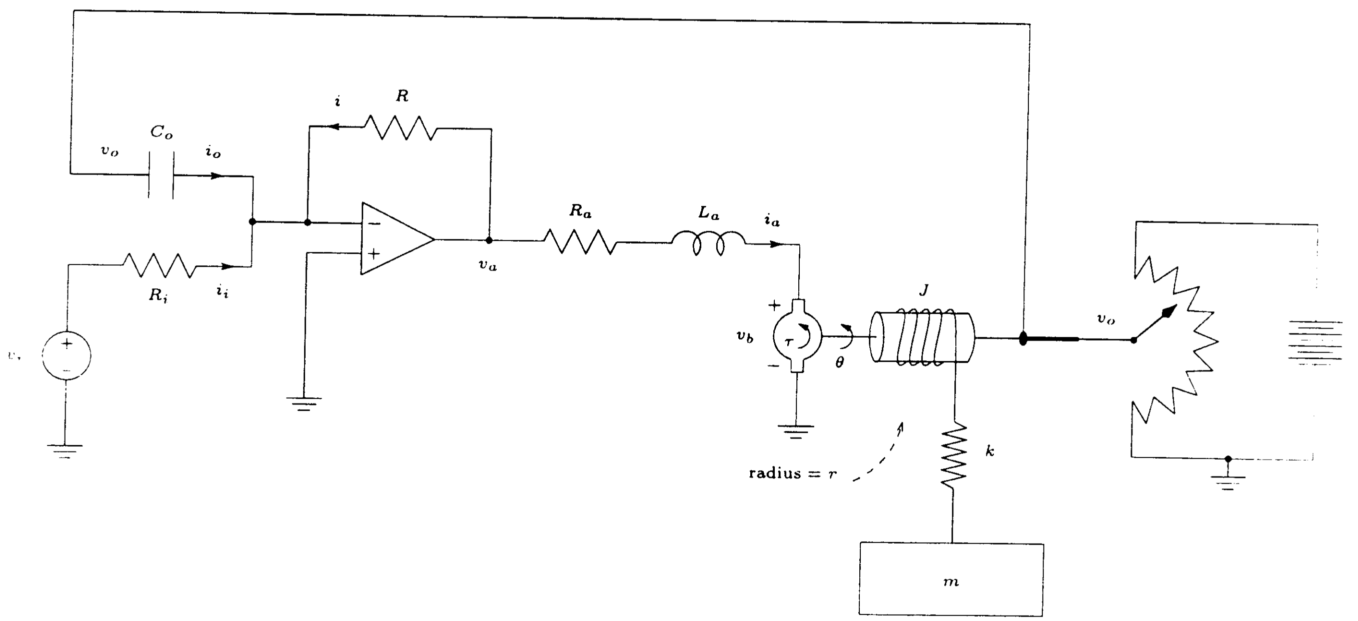
where u and $\boldsymbol{x}$ are the input and the state variables, respectively. Determine $\boldsymbol{x}(t)$ for $t \geq 0$; when $\boldsymbol{x}(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, and $u(t) = 1$ for $t \geq 0$. (25pts)

EE 231

Exam#1
Solutions

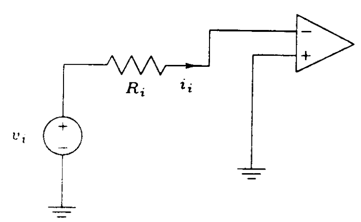
Oct. 18, 2005

1. The angular position of the shaft of a motor is controlled by the system shown below.



The angular position of the motor shaft is detected by a variable resistor which provides a voltage v_o proportional to the angle, such that $v_o = K_o\theta$. Draw the most detailed block diagram of the system, where v_i is the input, and θ is the output. Show all the variables v_i, i_i, v_o, i_o, v_a, i, i_a, v_b, τ, and θ as well as the displacement(s) associated with the mass-spring components on the block diagram.

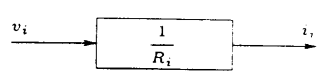
Solution: To determine the block diagram of the system, we first separate it into simpler components.

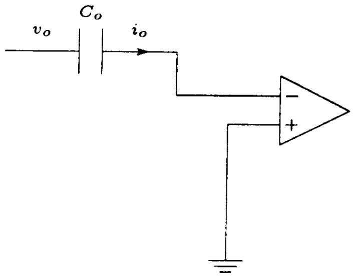


Since the input variable is v_i, we write i_i in terms v_i, such that

$$I_i(s) = \frac{1}{R_i} V_i(s),$$

since the operational amplifier is assumed to be ideal.



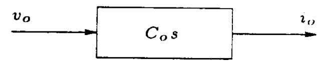


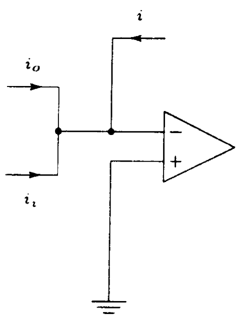
Similarly, we have

$$i_o(t) = C_o \frac{\mathrm{d}v_o(t)}{\mathrm{d}t},$$

or

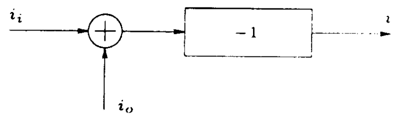
$$I_o(s) = C_o s V_o(s).$$

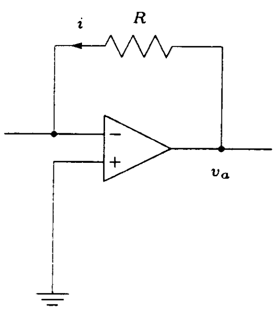




For an ideal operational amplifier,

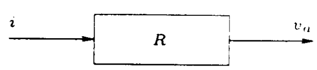
$$i(t) = -\big(i_i(t) + i_o(t)\big).$$

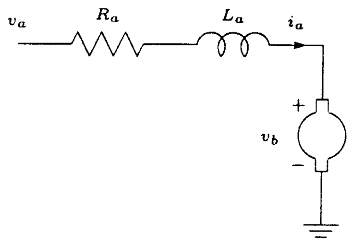




Again for an ideal operational amplifier,

$$V_a(s) = RI(s).$$



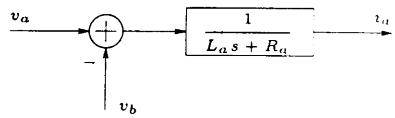


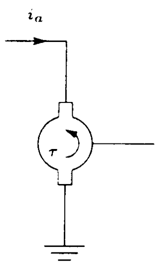
The armature current of the motor can be obtained from the Kirchhoff's Voltage Law, where

$$L_a \frac{\mathrm{d}i_a(t)}{\mathrm{d}t} + R_a i_a(t) + v_b(t) = v_a(t),$$

or

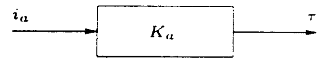
$$I_a(s) = \frac{1}{L_a s + R_a}\big(V_a(s) - V_b(s)\big).$$

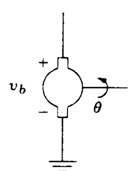




From the armature-controlled motor,

$$\tau(t) = K_a i_a(t).$$



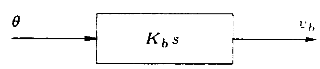


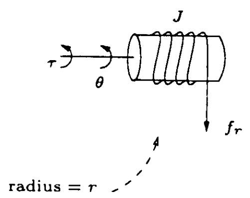
The back-emf voltage of the motor

$$v_b(t) = K_b \frac{\mathrm{d}\theta(t)}{\mathrm{d}t},$$

or

$$V_b(s) = (K_b s)\Theta(s).$$



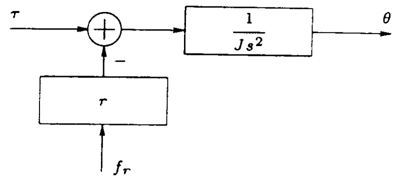


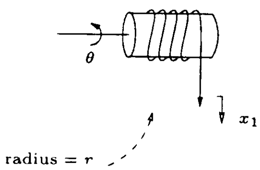
The torque equation for θ is

$$J\frac{\mathrm{d}^2\theta_m(t)}{\mathrm{d}t^2} = \tau(t) - rf_r(t),$$

where f_r is the internal tension of the rope. So,

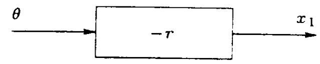
$$\Theta(s) = \frac{1}{Js^2}\big(T(s) - rF_r(s)\big).$$

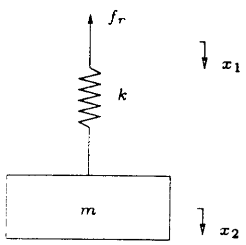




The disc with the inertia J changes the rotational motion to translational motion, where

$$x_1(t) = -r\theta(t).$$





The differential equations describing the the translational motion are

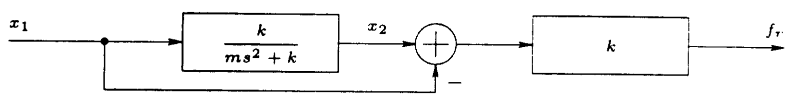
$$0 = -f_r - k(x_1 - x_2),$$

$$m\ddot{x}_2 = -k(x_2 - x_1).$$

So,

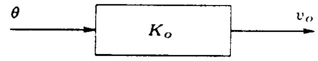
$$F_r(s) = k\big(X_2(s) - X_1(s)\big),$$

$$X_2(s) = \frac{k}{ms^2 + k}X_1(s).$$

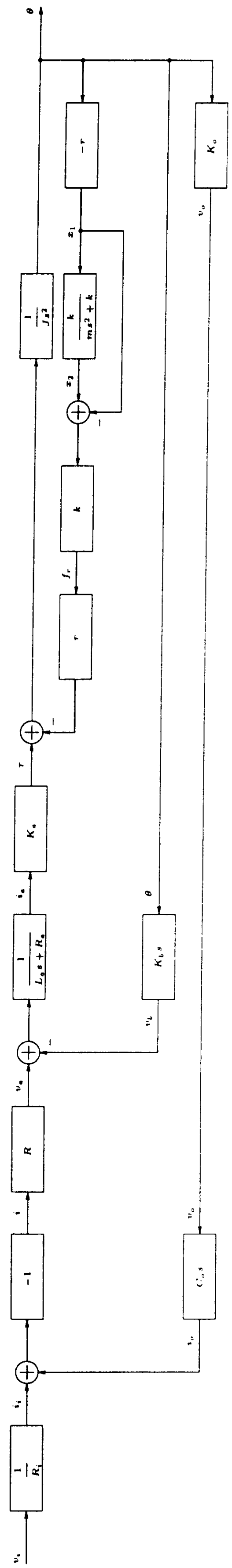


And, finally the given relationship

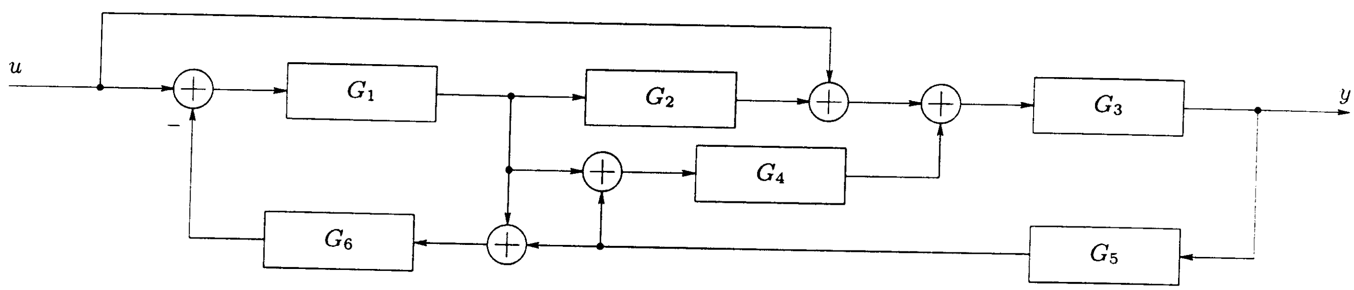
$$v_o(t) = K_o\theta(t).$$



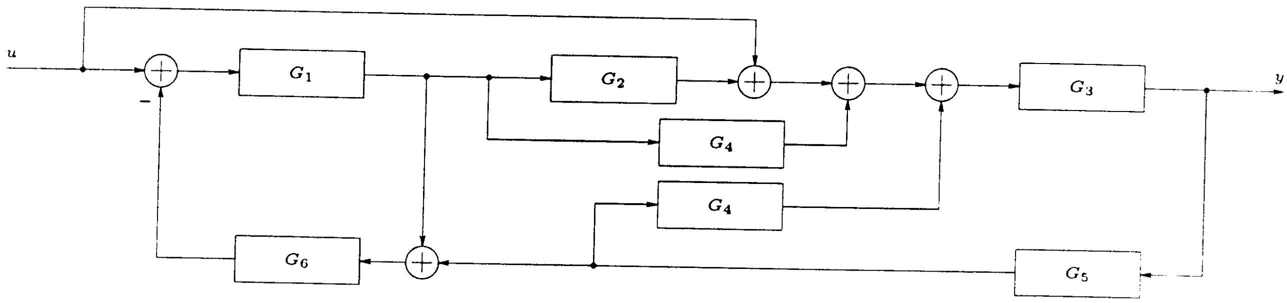
When we connect all the individual blocks together, we get the following block diagram.

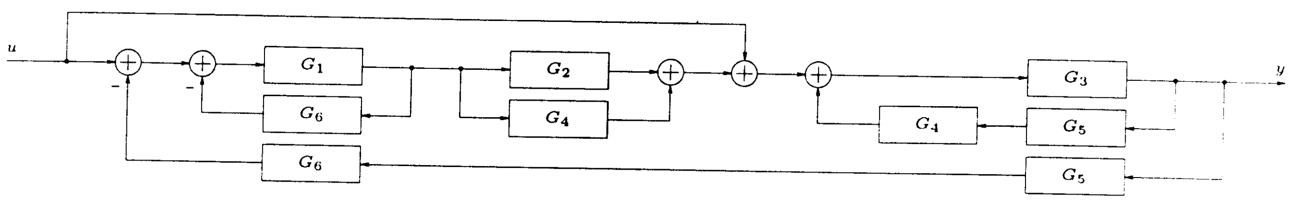

v_s
1/R_1
i_1
-1
i
R
v_a
1/(L_a s + R_a)
i_a
K_a
τ
r
f_r
k
x_2
k/(ms^2 + k)
x_1
-r
1/(Js^2)
θ
K_o
v_o
C_o s
i_o
K_b s
v_b

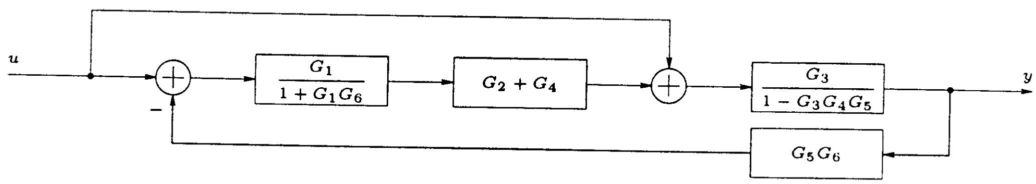
2. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly.

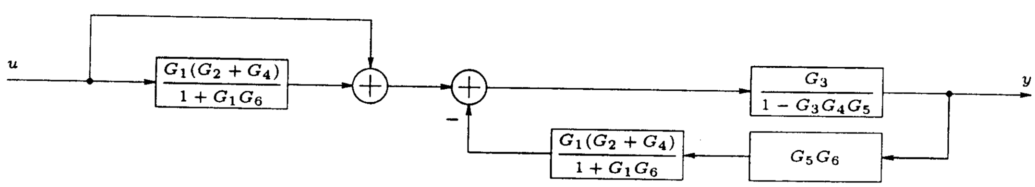


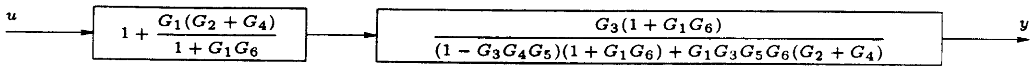
Solution: If we choose to use the block-diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.

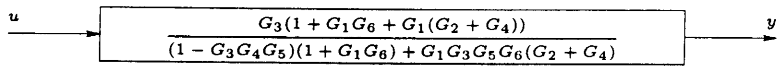




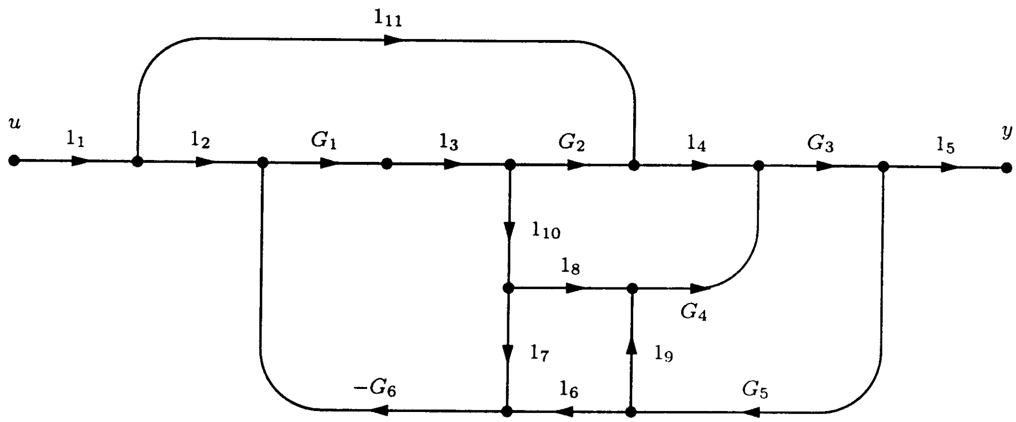








If we choose to use Mason's formula, we need to draw the signal flow graph of the block diagram.



In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The forward path gains are

$$F_1 = 1_1 1_2 G_1 1_3 G_2 1_4 G_3 1_5 = G_1 G_2 G_3,$$

$$F_2 = 1_1 1_{11} 1_4 G_3 1_5 = G_3,$$

$$F_3 = 1_1 1_2 G_1 1_3 1_{10} 1_8 G_4 G_3 1_5 = G_1 G_3 G_4.$$

The loop gains are

$$L_1 = G_1 1_3 1_{10} 1_7 (-G_6) = -G_1 G_6,$$

$$L_2 = G_1 1_3 1_{10} 1_8 G_4 G_3 G_5 1_6 (-G_6) = -G_1 G_3 G_4 G_5 G_6,$$

$$L_3 = G_1 1_3 G_2 1_4 G_3 G_5 1_6 (-G_6) = -G_1 G_2 G_3 G_5 G_6,$$

$$L_4 = G_3 G_5 1_9 G_4 = G_3 G_4 G_5.$$

From the forward path and the loop gains, we determine the touching loops and the forward paths.

Touching Loops

	L_1	L_2	L_3	L_4
L_1	✔	✔	✔	✘
L_2		✔	✔	✔
L_3			✔	✔
L_4				✔

Loops on Forward Paths

	L_1	L_2	L_3	L_4
F_1	✔	✔	✔	✔
F_2	✘	✔	✔	✔
F_3	✔	✔	✔	✔

Therefore,

$$\begin{aligned}\Delta &= 1 - \big(L_1 + L_2 + L_3 + L_4\big) + \big(L_1 L_4\big)\\ &= 1 - \big((-G_1G_6) + (-G_1G_3G_4G_5G_6) + (-G_1G_2G_3G_5G_6) + (G_3G_4G_5)\big) + \big((-G_1G_6)(G_3G_4G_5)\big)\\ &= 1 + G_1G_6 + G_1G_3G_4G_5G_6 + G_1G_2G_3G_5G_6 - G_3G_4G_5 - G_1G_3G_4G_5G_6,\\ &= 1 + G_1G_6 + G_1G_2G_3G_5G_6 - G_3G_4G_5,\end{aligned}$$

and

$$\begin{aligned}\Delta_1 &= \Delta|_{L_1=L_2=L_3=L_4=0} = 1,\\ \Delta_2 &= \Delta|_{L_2=L_3=L_4=0} = 1 - L_1 = 1 + G_1G_6,\\ \Delta_3 &= \Delta|_{L_1=L_2=L_3=L_4=0} = 1.\end{aligned}$$

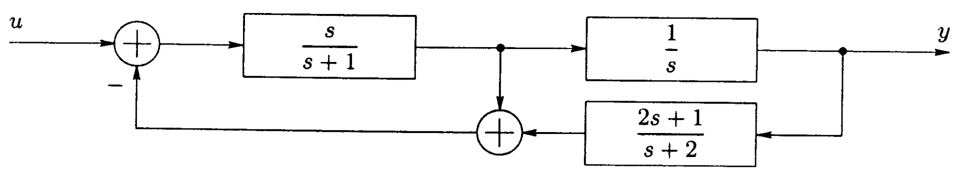
So,

$$\frac{Y(s)}{U(s)} = \frac{1}{\Delta}\sum_{i=1}^{3} F_i\Delta_i = \frac{(G_1G_2G_3)(1) + (G_3)(1 + G_1G_6) + (G_1G_3G_4)(1)}{1 + G_1G_6 + G_1G_2G_3G_5G_6 - G_3G_4G_5},$$

or

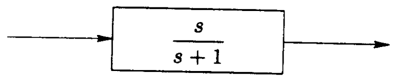
$$\frac{Y(s)}{U(s)} = \frac{G_1G_2G_3 + G_3(1 + G_1G_6) + G_1G_3G_4}{1 + G_1G_6 + G_1G_2G_3G_5G_6 - G_3G_4G_5}.$$

3. The block diagram of a control system is given below.

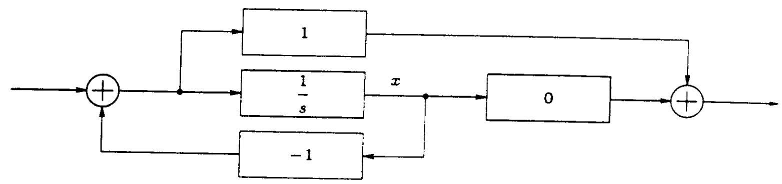


Obtain a state-space representation of the system without any block-diagram reduction.

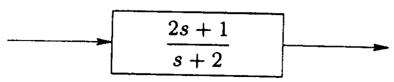
Solution: In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.



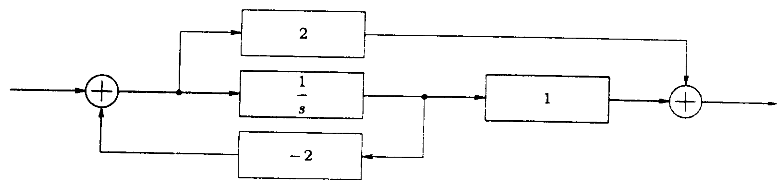
(a) The first feedforward gain block.



(b) Controller realization form.

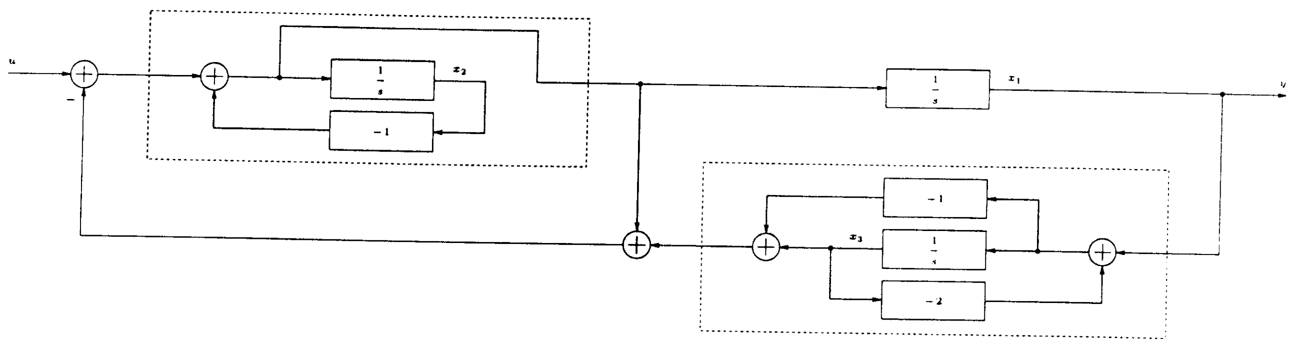


(c) The feedback gain block.



(d) Controller realization form.

The connected and "expanded" block diagram is shown below.



After assigning the state variables as shown in the figure, we obtain

$$\dot{x}_1 = \dot{x}_2,$$

$$\dot{x}_2 = -x_2 + \Big(u - \big(\dot{x}_2 + (2\dot{x}_3 + x_3)\big)\Big),$$

$$\dot{x}_3 = -2x_3 + x_1,$$

and

$$y = x_1.$$

From the second state equation, we get

$$2\dot{x}_2 = -x_2 + u - 2\left(-2x_3 + x_1\right) - x_3,$$

or

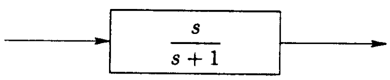
$$\dot{x}_2 = -x_1 - (1/2)x_2 + (3/2)x_3 + (1/2)u.$$

After substituting the above equation into the original set, we obtain the state-space representation

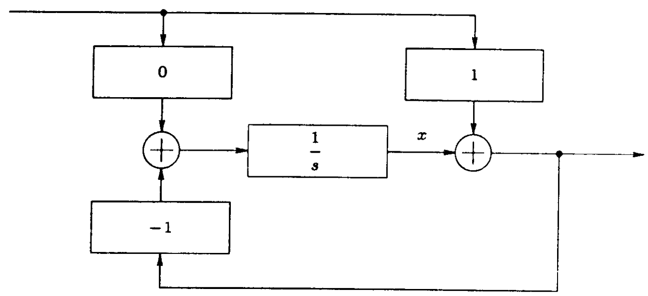
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -1 & -1/2 & 3/2 \\ -1 & -1/2 & 3/2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

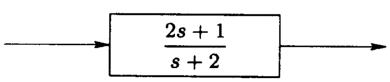
If we use the observer realization form for each of the blocks, then we obtain a different state-space representation.



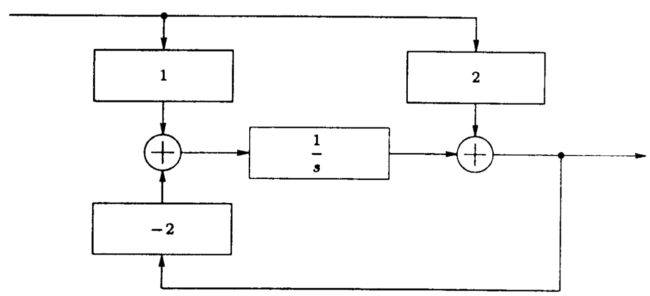
(a) The first feedforward gain block.



(b) Observer realization form.

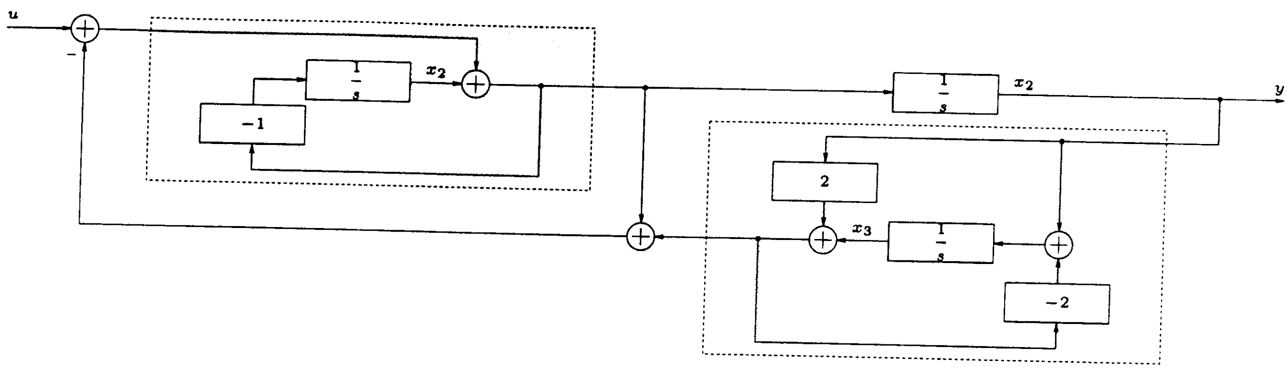


(c) The feedback gain block.



(d) Observer realization form.

The connected and "expanded" block diagram for this case is shown below.



Similarly, we obtain

$$\dot{x}_1 = x_2 + \Big(u - \big(\dot{x}_1 + (x_3 + 2x_1)\big)\Big),$$

$$\dot{x}_2 = -\dot{x}_1,$$

$$\dot{x}_3 = x_1 - 2\,(x_3 + 2x_1),$$

and

$$y = x_1.$$

From the first state equation, we get

$$2\dot{x}_1 = -2x_1 + x_2 - x_3 + u,$$

or

$$\dot{x}_1 = -x_1 + (1/2)x_2 - (1/2)x_3 + (1/2)u.$$

And,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 1/2 & -1/2 \\ 1 & -1/2 & 1/2 \\ -3 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

4. The state equation of a control system is given by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

where u and $\boldsymbol{x}$ are the input and the state variables, respectively. Determine $\boldsymbol{x}(t)$ for $t \geq 0$; when $\boldsymbol{x}(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, and $u(t) = 1$ for $t \geq 0$.

Solution: The general solution to the state equation of a system described by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

is obtained from

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)}B\boldsymbol{u}(\tau)\,\mathrm{d}\tau,$$

where

$$e^{At} = \mathcal{L}_s^{-1}\left[(sI - A)^{-1}\right](t).$$

Here, I is the appropriately dimensioned identity matrix. In our case,

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$\boldsymbol{x}(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, and $u(t) = 1$ for $t \geq 0$. We first need to determine the state-transition matrix

$$e^{At} = \mathcal{L}_s^{-1}\left[(sI - A)^{-1}\right](t) = \mathcal{L}_s^{-1}\left[\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\right)^{-1}\right](t)$$

$$= \mathcal{L}_s^{-1}\left[\left(\begin{bmatrix} s & -2 \\ 2 & s \end{bmatrix}\right)^{-1}\right](t) = \mathcal{L}_s^{-1}\left[\frac{1}{s^2+2^2}\begin{bmatrix} s & 2 \\ -2 & s \end{bmatrix}\right](t)$$

$$= \begin{bmatrix} \mathcal{L}_s^{-1}\left[\frac{s}{s^2+2^2}\right](t) & \mathcal{L}_s^{-1}\left[\frac{2}{s^2+2^2}\right](t) \\ \mathcal{L}_s^{-1}\left[\frac{-2}{s^2+2^2}\right](t) & \mathcal{L}_s^{-1}\left[\frac{s}{s^2+2^2}\right](t) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix}.$$

From the state-transition matrix, we can determine the state variable

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau$$

$$= \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \int_0^t \begin{bmatrix} \cos(2t-2\tau) & \sin(2t-2\tau) \\ -\sin(2t-2\tau) & \cos(2t-2\tau) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(1\right) \mathrm{d}\tau$$

$$= \begin{bmatrix} \cos(2t) - \sin(2t) \\ -\sin(2t) - \cos(2t) \end{bmatrix} + \int_0^t \begin{bmatrix} \sin(2t-2\tau) \\ \cos(2t-2\tau) \end{bmatrix} \mathrm{d}\tau$$

$$= \begin{bmatrix} \cos(2t) - \sin(2t) \\ -\cos(2t) - \sin(2t) \end{bmatrix} + \begin{bmatrix} (1/2)\cos(2t-2\tau) \\ -(1/2)\sin(2t-2\tau) \end{bmatrix}_{\tau=0}^{\tau=t}$$

$$= \begin{bmatrix} \cos(2t) - \sin(2t) \\ -\cos(2t) - \sin(2t) \end{bmatrix} + \begin{bmatrix} (1/2) - (1/2)\cos(2t) \\ (1/2)\sin(2t) \end{bmatrix}$$

Or,

$$\boldsymbol{x}(t) = \begin{bmatrix} (1/2) + (1/2)\cos(2t) - \sin(2t) \\ -\cos(2t) - (1/2)\sin(2t) \end{bmatrix} \text{ for } t \geq 0.$$