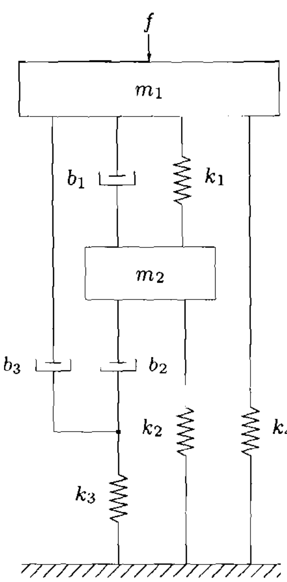
EE 231 **Exam#1** **Oct. 16, 2007**

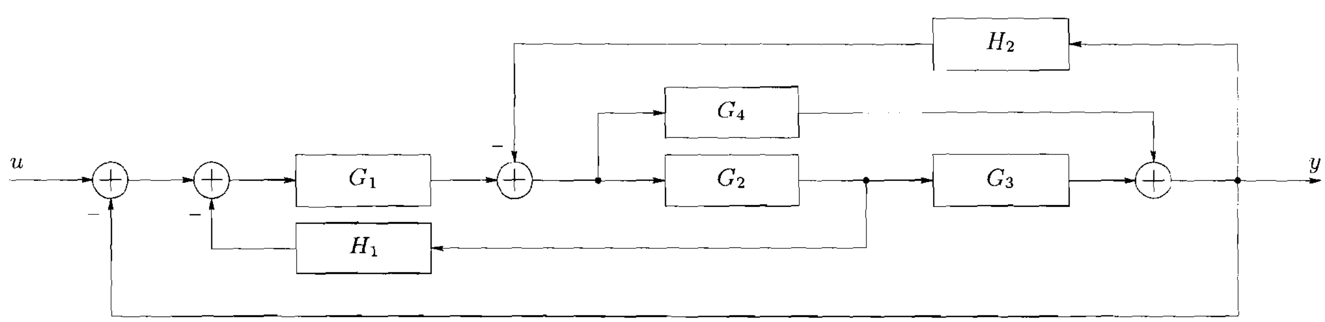
75 minutes

1. Consider the following mechanical system.



Obtain the force-current equivalent of the system. (20pts)

2. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly. (30pts)



3. A nonlinear amplifier with an input e_i and an output e_o can be described by

$$e_o(t) = \begin{cases} -e_i^3(t) + e_i(t), & \text{if } e_i(t) < 0; \\ e_i^3(t) + e_i(t), & \text{if } e_i(t) \geq 0. \end{cases}$$

Obtain the affine approximation of the output about the following operating points, and sketch the nonlinear function with its approximations.

(a) $e_i = 0$. (05pts)

(b) $e_i = 2$. (10pts)

4. A control system is described in state-space representation, such that

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} -2 & 1 \end{bmatrix} \boldsymbol{x}(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively. Determine the transfer function or the transfer matrix of the system. (15pts)

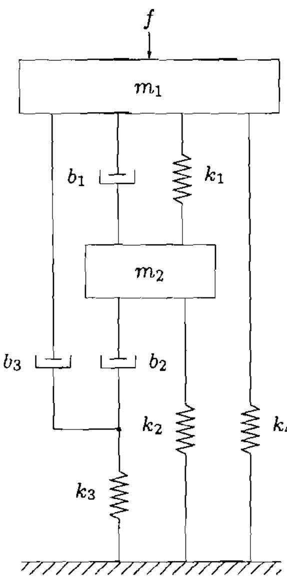
5. The following requirements are given for a second-order system that is described by the transfer function $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$.

$$\begin{aligned} \text{Maximum percent overshoot:} &\quad M_p \geq 20\%. \\ 5\% \text{ settling time:} &\quad 5\,\text{s} \leq t_{5\%s} \leq 10\,\text{s}. \end{aligned}$$

(a) Describe and sketch the s-plane regions of the pole locations satisfying the requirements. (10pts)

(b) Determine the largest possible peak time of a system with the poles satisfying the requirements. (10pts)

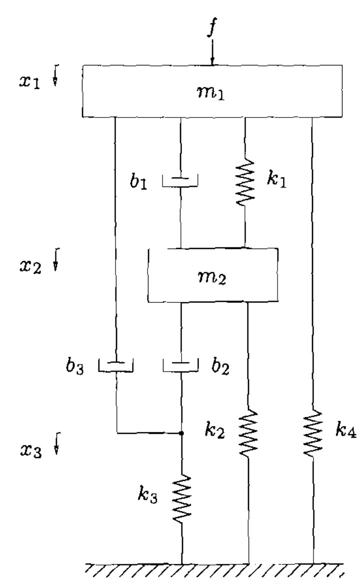
1. Consider the following mechanical system.



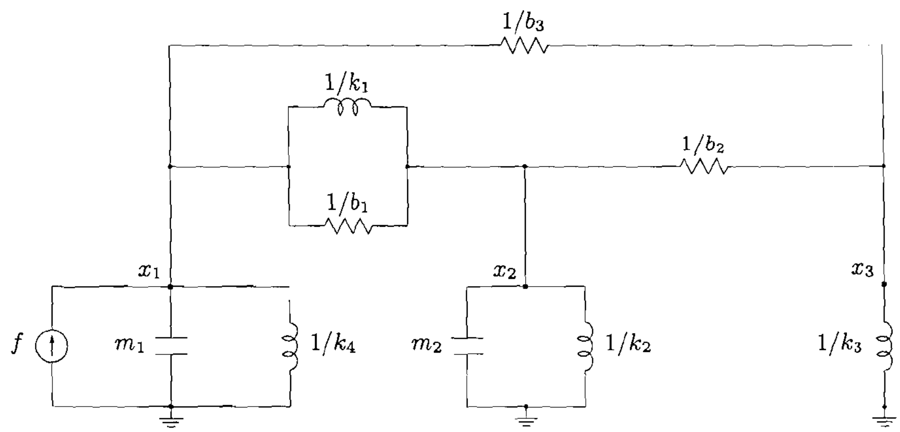
Obtain the force-current equivalent of the system.

Solution:

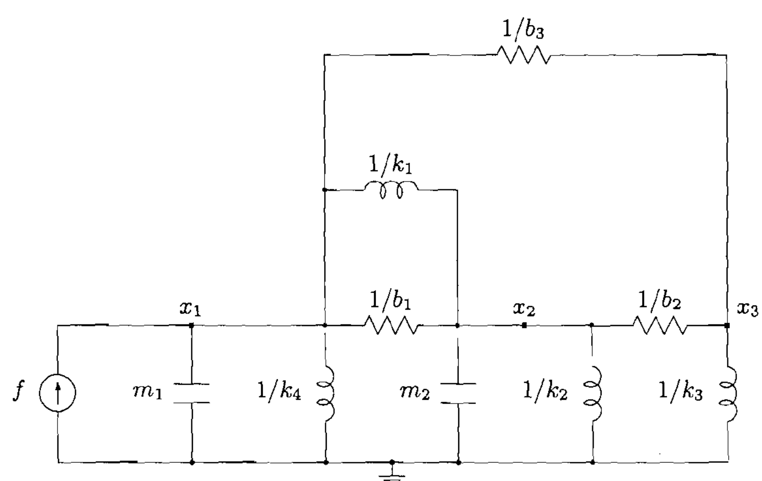
First, we identify the linearly independent displacement locations in the mechanical system and mark them.



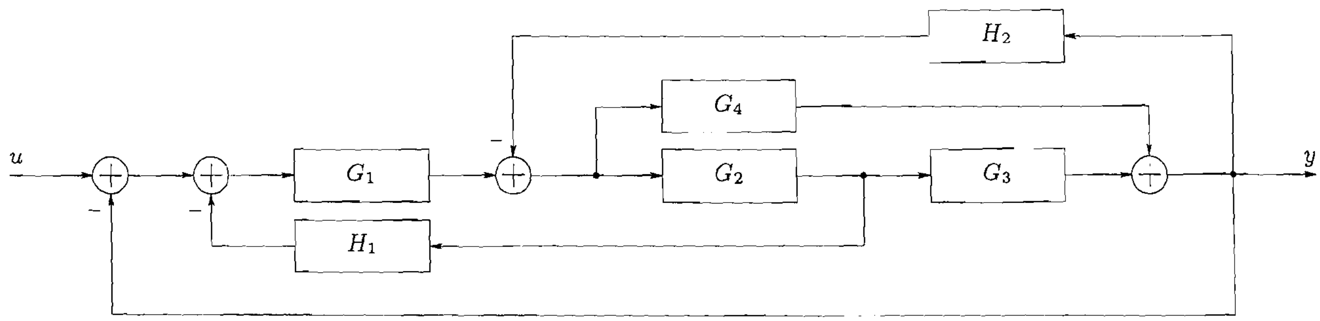
For the force-current analog of a mechanical system, there will be a node flux associated with each displacement variable (or a node voltage associated with each velocity variable), and an input force will be associated with a current source. The spring constant, the damping constant, and the mass will be associated with the reciprocal of inductance, the conductance, and the capacitance, respectively. The elements between two displacement variables of the mechanical system will be between the corresponding node variables of the force-current analog. The elements that are connected to fixed frames and the elements that are always measured with respect to a fixed frame, such as the mass and the external force, will be connected to the ground.



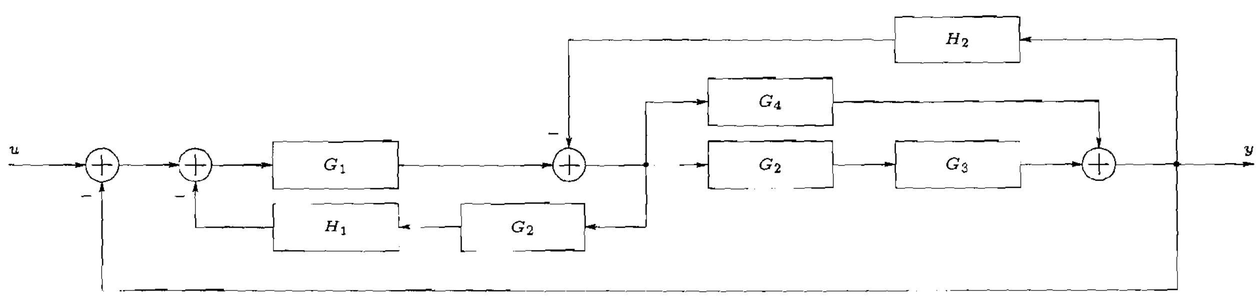
Or, drawing the circuit diagram more compact, we get the following diagram.

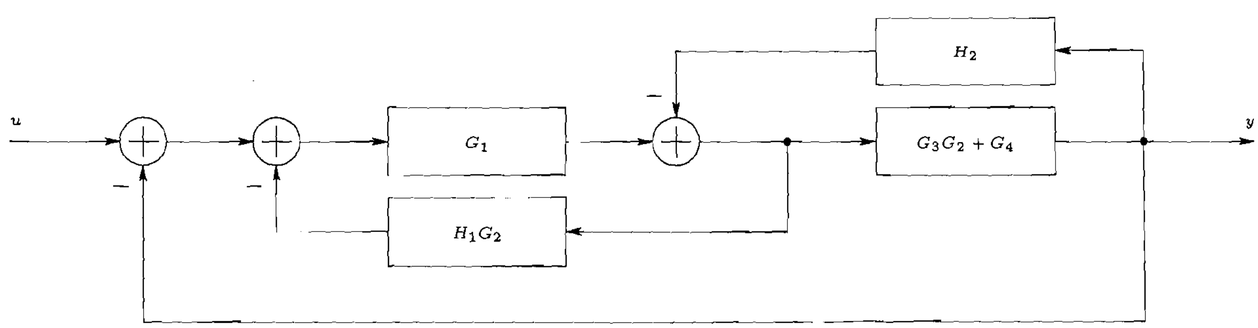


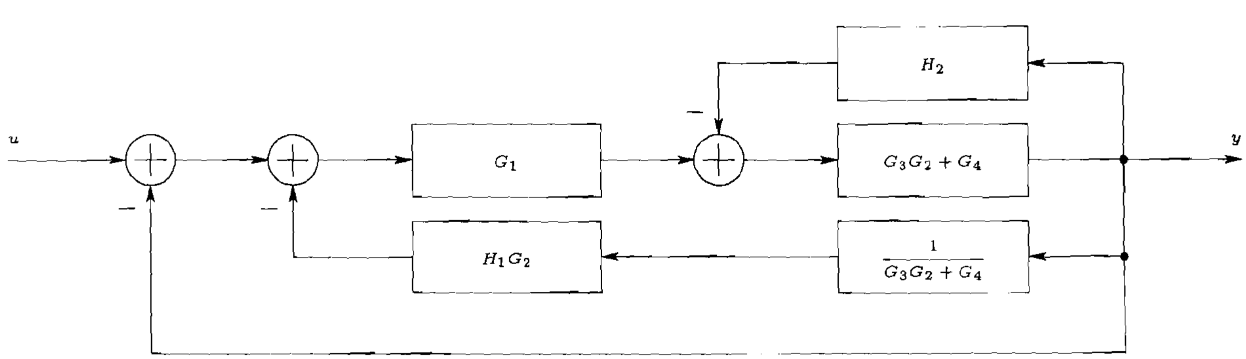
2. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly.

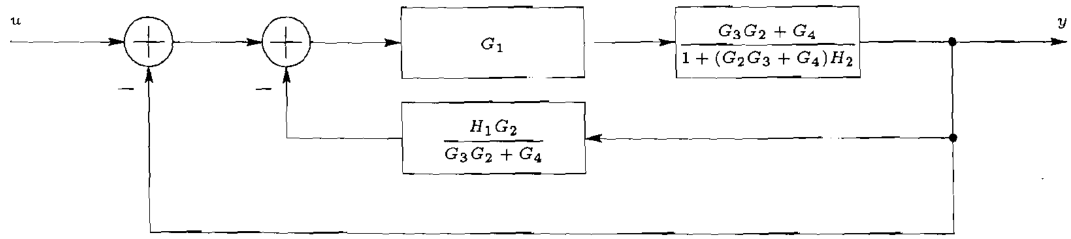


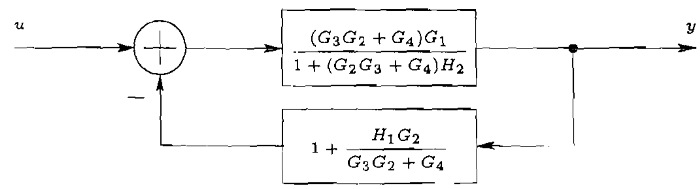
Solution: If we choose to use the block-diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.

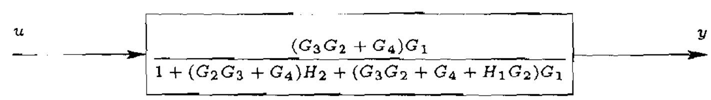




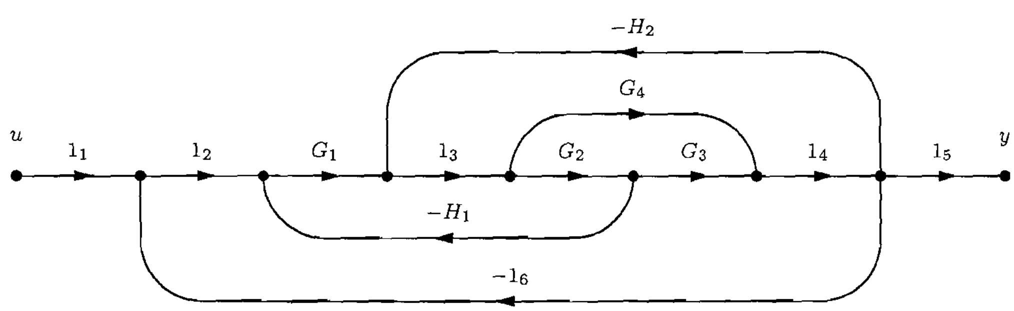








If we choose to use Mason's formula, we need to draw the signal flow graph of the block diagram.



In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The forward path gains are

$$F_1 = 1_1 1_2 G_1 1_3 G_2 G_3 1_4 1_5 = G_1 G_2 G_3,$$

$$F_2 = 1_1 1_2 G_1 1_3 G_4 1_4 1_5 = G_1 G_4.$$

The loop gains are

$$L_1 = 1_2 G_1 1_3 G_2 G_3 1_4 (-1_6) = -G_1 G_2 G_3,$$

$$L_2 = 1_2 G_1 1_3 G_4 1_4 (-1_6) = -G_1 G_4,$$

$$L_3 = G_1 1_3 G_2 (-H_1) = -G_1 G_2 H_1,$$

$$L_4 = 1_3 G_2 G_3 1_4 (-H_2) = -G_2 G_3 H_2,$$

$$L_5 = 1_3 G_4 1_4 (-H_2) = -G_4 H_2.$$

From the forward path and the loop gains, we determine the touching loops and the forward paths.

Touching Loops

	L_1	L_2	L_3	L_4	L_5
L_1	✔	✔	✔	✔	✔
L_2		✔	✔	✔	✔
L_3			✔	✔	✔
L_4				✔	✔
L_5					✔

Loops on Forward Paths

	L_1	L_2	L_3	L_4	L_5
F_1	✔	✔	✔	✔	✔
F_2	✔	✔	✔	✔	✔

Therefore,

$$\begin{aligned}\Delta &= 1 - \big(L_1 + L_2 + L_3 + L_4 + L_5\big)\\ &= 1 - \big((-G_1G_2G_3) + (-G_1G_4) + (-G_1G_2H_1) + (-G_2G_3H_2) + (-G_4H_2)\big)\\ &= 1 + G_1G_2G_3 + G_1G_4 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2,\end{aligned}$$

and

$$\begin{aligned}\Delta_1 &= \Delta|_{L_1=L_2=L_3=L_4=L_5=0} = 1,\\ \Delta_2 &= \Delta|_{L_1=L_2=L_3=L_4=L_5=0} = 1.\end{aligned}$$

So,

$$\frac{Y(s)}{U(s)} = \frac{1}{\Delta}\sum_{i=1}^{2} F_i\Delta_i = \frac{(G_1G_2G_3)(1) + (G_1G_4)(1)}{1 + G_1G_2G_3 + G_1G_4 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2},$$

or

$$\frac{Y(s)}{U(s)} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2G_3 + G_1G_4 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2}.$$

3. A nonlinear amplifier with an input e_i and an output e_o can be described by

$$e_o(t) = \begin{cases} -{e_i}^3(t) + e_i(t), & \text{if } e_i(t) < 0;\\ {e_i}^3(t) + e_i(t), & \text{if } e_i(t) \geq 0.\end{cases}$$

Obtain the affine approximation of the output about the following operating points, and sketch the nonlinear function with its approximations.

(a) $e_i = 0$.

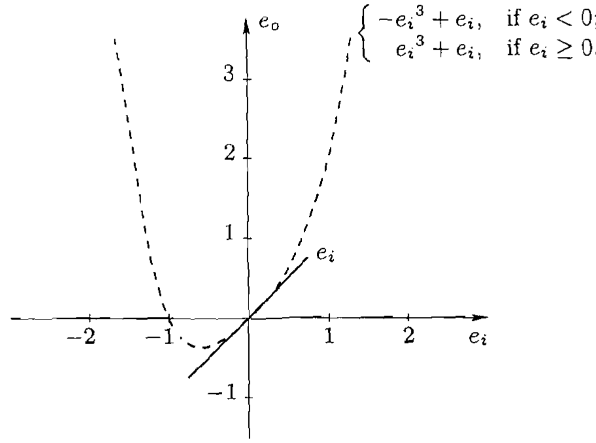
Solution: The affine approximation is obtained by keeping the constant and the linear terms in the taylor series expansion about the operating point of the nonlinear terms. In our case,

$$\begin{aligned} {e_i}^3 &= \left[{e_i}^3\right]_{e_i=0} + \left[\frac{\mathrm{d}}{\mathrm{d}e_i}\left({e_i}^3\right)\right]_{e_i=0}\left(e_i - 0\right) + \mathcal{O}({e_i}^2) \\ &= \left(0\right) + \left[3{e_i}^2\right]_{e_i=0} e_i + \mathcal{O}({e_i}^2) = \left(0\right)e_i + \mathcal{O}({e_i}^2) \\ &\approx 0. \end{aligned}$$

So,

$$e_o(t) = \begin{cases} -{e_i}^3(t) + e_i(t), & \text{if } e_i(t) < 0; \\ {e_i}^3(t) + e_i(t), & \text{if } e_i(t) \geq 0; \end{cases} \approx \begin{cases} e_i(t), & \text{if } e_i(t) < 0 \text{ and } e_i(t) \approx 0; \\ e_i(t), & \text{if } e_i(t) \geq 0 \text{ and } e_i(t) \approx 0. \end{cases}$$

$$\begin{aligned} e_o(t) &= \begin{cases} -{e_i}^3(t) + e_i(t), & \text{if } e_i(t) < 0; \\ {e_i}^3(t) + e_i(t), & \text{if } e_i(t) \geq 0; \end{cases} \\ &\approx e_i(t) \text{ for } e_i(t) \approx 0. \end{aligned}$$



(b) $e_i = 2$.

Solution: The affine approximation is obtained by keeping the constant and the linear terms in the taylor series expansion about the operating point of the nonlinear terms. In our case,

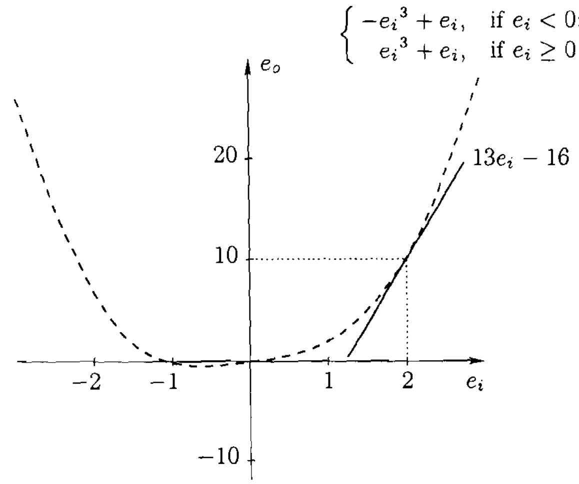
$$\begin{aligned} {e_i}^3 &= \left[{e_i}^3\right]_{e_i=2} + \left[\frac{\mathrm{d}}{\mathrm{d}e_i}\left({e_i}^3\right)\right]_{e_i=2}\left(e_i - 2\right) + \mathcal{O}({e_i}^2) \\ &= \left(2^3\right) + \left[3{e_i}^2\right]_{e_i=2}\left(e_i - 2\right) + \mathcal{O}({e_i}^2) = 8 + 12\left(e_i - 2\right) + \mathcal{O}({e_i}^2) \\ &\approx 12e_i - 16. \end{aligned}$$

So,

$$\begin{aligned} e_o(t) &= \begin{cases} -{e_i}^3(t) + e_i(t), & \text{if } e_i(t) < 0; \\ {e_i}^3(t) + e_i(t), & \text{if } e_i(t) \geq 0. \end{cases} \\ &\approx \left(12e_i(t) - 16\right) + e_i(t) = 13e_i(t) - 16, \text{ if } e_i(t) \approx 2. \end{aligned}$$

$$e_o(t) = \begin{cases} -e_i^3(t) + e_i(t), & \text{if } e_i(t) < 0; \\ e_i^3(t) + e_i(t), & \text{if } e_i(t) \geq 0; \end{cases}$$

$$\approx 13e_i(t) - 16 \text{ for } e_i(t) \approx 2.$$



4. A control system is described in state-space representation, such that

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} -2 & 1 \end{bmatrix} \boldsymbol{x}(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively. Determine the transfer function or the transfer matrix of the system.

Solution: The transfer matrix of a control system described in the state-state representation

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$

$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t),$$

is

$$F(s) = C(sI - A)^{-1}B + D,$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$C = \begin{bmatrix} -2 & 1 \end{bmatrix}, \qquad D = 0,$$

and I is the appropriately dimensioned identity matrix. So,

$$F(s) = \begin{bmatrix} -2 & 1 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0$$

$$= \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{(s+2)^2} \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{(s+2)^2} \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} s+2 \\ 2(s+2) \end{bmatrix}$$

$$= \frac{1}{(s+2)^2} (0).$$

In other words, the transfer function is $F(s) = 0$.

5. The following requirements are given for a second-order system that is described by the transfer function $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$.

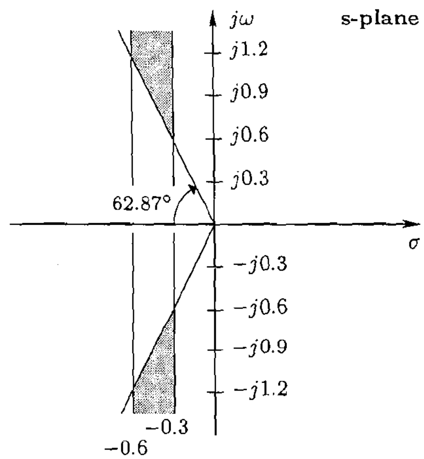
Maximum percent overshoot: $M_p \geq 20\%$.
5% settling time: $5\,\mathrm{s} \leq t_{5\%s} \leq 10\,\mathrm{s}$.

(a) Describe and sketch the s-plane regions of the pole locations satisfying the requirements.

Solution:

Given Specifications	System Constraints	Geometrical Representations
$20\% \leq M_p$.	$0.20 \leq e^{-\left(\zeta/\sqrt{1-\zeta^2}\right)\pi}$, $\zeta \leq \frac{\lvert\ln(0.20)\rvert}{\sqrt{(\ln(0.20))^2 + (\pi)^2}}$, or $\zeta \leq 0.46$; since $M_p = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$, and $\zeta = \lvert\ln(M_p)\rvert/\sqrt{(\ln(M_p))^2 + (\pi)^2}$.	$\cos^{-1}(0.46) \leq \alpha$ or $62.87^\circ \leq \alpha$, where $\alpha = \cos^{-1}(\zeta)$ is the angle measured from the negative real axis.
$5\,\mathrm{s} \leq t_{5\%s} \leq 10\,\mathrm{s}$.	$5 \leq \frac{3}{\sigma_o} \leq 10$, or $3/5 \geq \sigma_o \geq 3/10$; since $t_{5\%s} = 3/\sigma_o$.	$-0.6 \leq \sigma \leq -0.3$, since the poles are at $s = -\sigma_o \pm j\omega_d$

The shaded region describes the region specified by the given requirements.



(b) Determine the largest possible peak time of a system with the poles satisfying the requirements.

Solution: The peak time of the system is given by

$$t_p = \frac{\pi}{\omega_d}.$$

The largest peak time is when we have the smallest ω_d. From the shaded region of the sketch in the previous part, we realize that the smallest ω_d is when $\omega_d = 0.3\tan(62.87°) \approx 0.59$, which is at the intersection of the radial line with the angle of 62.87° with respect to the negative real axis and the vertical line at $\sigma = -0.3$. Therefore,

$$t_{p_{\max}} = \frac{\pi}{\omega_{d_{\min}}} = \frac{\pi}{0.59};$$

or the largest possible peak time of the system is 5.36 s.