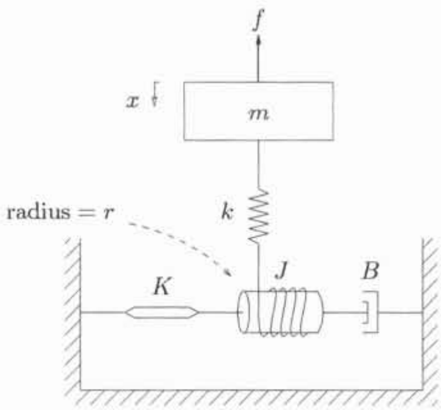
EE 231 | **Exam#1** | **Mar. 19, 2009**

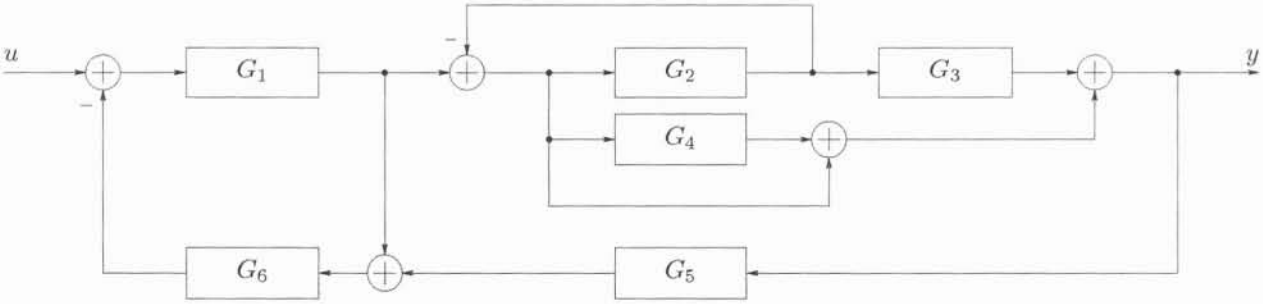
75 minutes

1. Consider the following mechanical system.



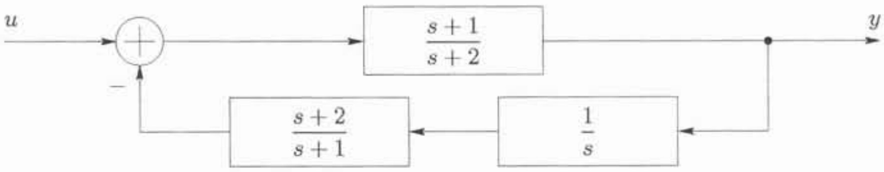
Obtain the detailed block diagram of the system assuming f is the input and x is the output. Show all the internal displacement(s) and angle(s). (25pts)

2. Consider the following block diagram.



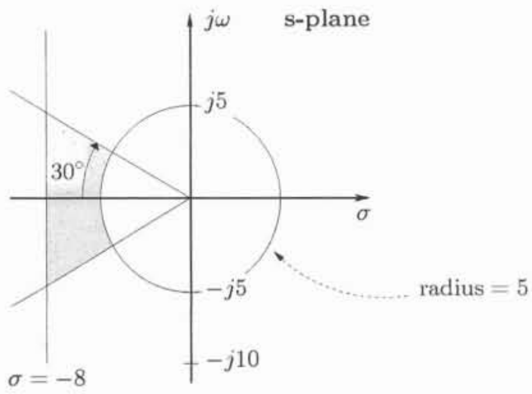
Determine the transfer function. Show your work clearly. (25pts)

3. The block diagram of a control system is given below.



Obtain a state-space representation of the system describing all the internal dynamics. (25pts)

4. Consider a second-order system described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, such that its poles are located in the shaded region below.



Obtain the necessary inequalities to describe the poles in the shaded region below in terms of only ζ and ω_n. (25pts)

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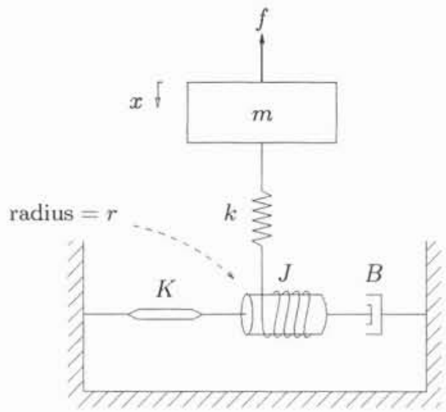
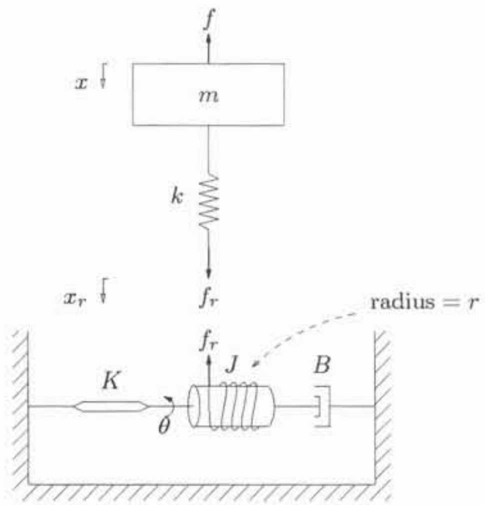
Solutions

1. Consider the following mechanical system.

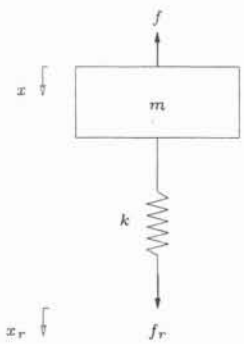


Obtain the detailed block diagram of the system assuming f is the input and x is the output. Show all the internal displacement(s) and angle(s).

Solution: First, we identify the linearly independent linear and rotational displacement locations in the mechanical system and mark them. At this point, we also separate the linear and rotational components.



Then, we write the differential equations describing the linear and the rotational motions.

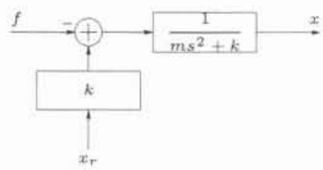


The translational portion has two equations, one for each of the displacements.

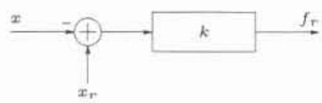
$$m\ddot{x} = -f - k(x - x_r).$$
$$0 = f_r - k(x_r - x).$$

After taking the laplace transforms of the equations under zero initial conditions and rearranging the variables, we have the following block diagrams.

$$X(s) = \left(\frac{1}{ms^2 + k}\right)\big(kX_r(s) - F(s)\big).$$

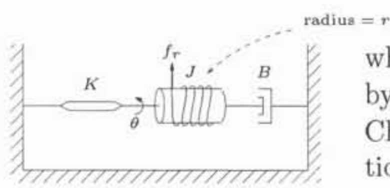


$$F_r(s) = k\big(X_r(s) - X(s)\big).$$



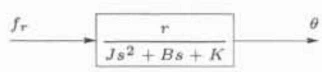
There is one angle at the rotational portion.

$$J\ddot{\theta} = rf_r - B\dot{\theta} - K\theta,$$



where rf_r is the torque generated by the linear portion of the system. Choosing θ as the output of this portion, we get

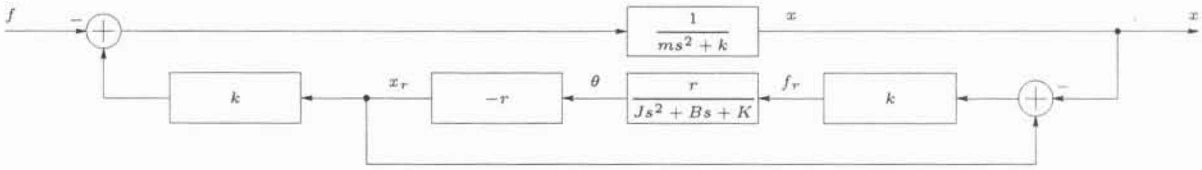
$$\Theta(s) = \frac{r}{Js^2 + Bs + K}F_r(s).$$



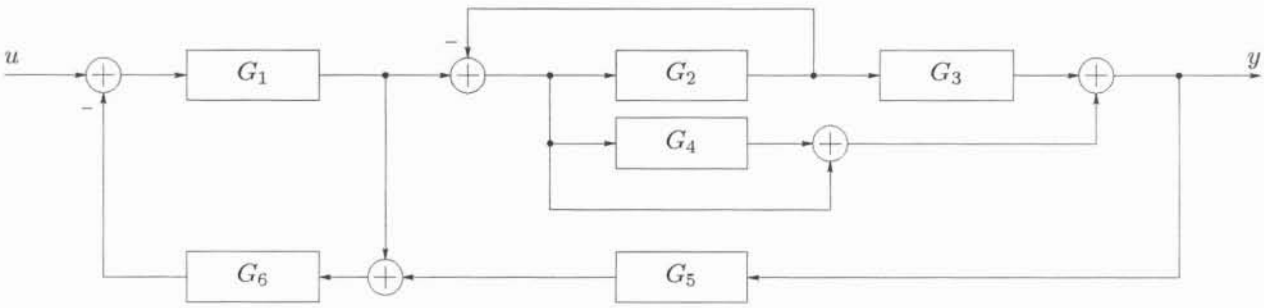
The displacement x_r is set by the rotation of the bottom rotational portion, where

$$x_r = -r\theta.$$

When we connect all the individual blocks together, we get the following block diagram.

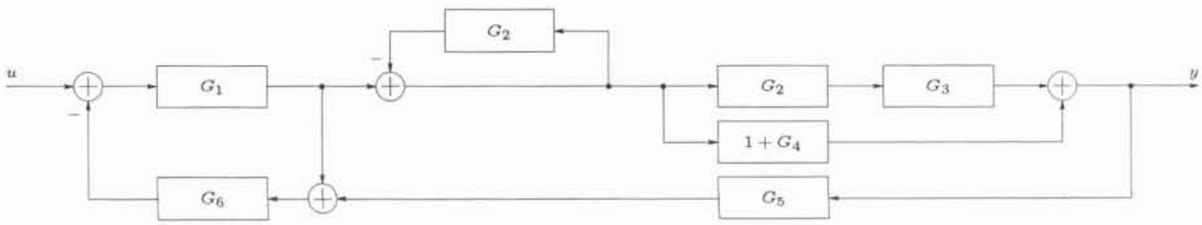


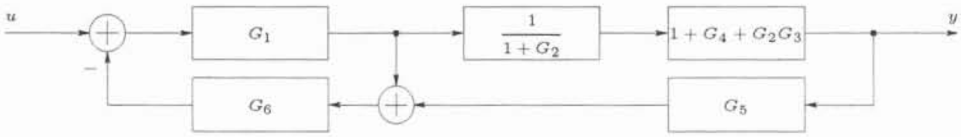
2. Consider the following block diagram.

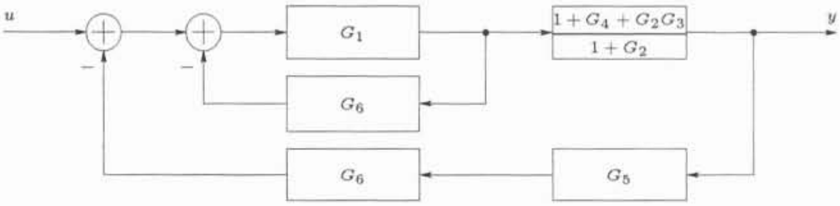


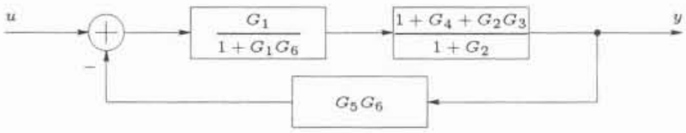
Determine the transfer function. Show your work clearly.

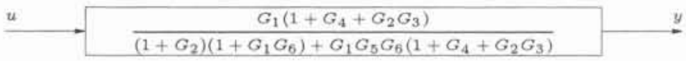
Solution: If we choose to use the block-diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.



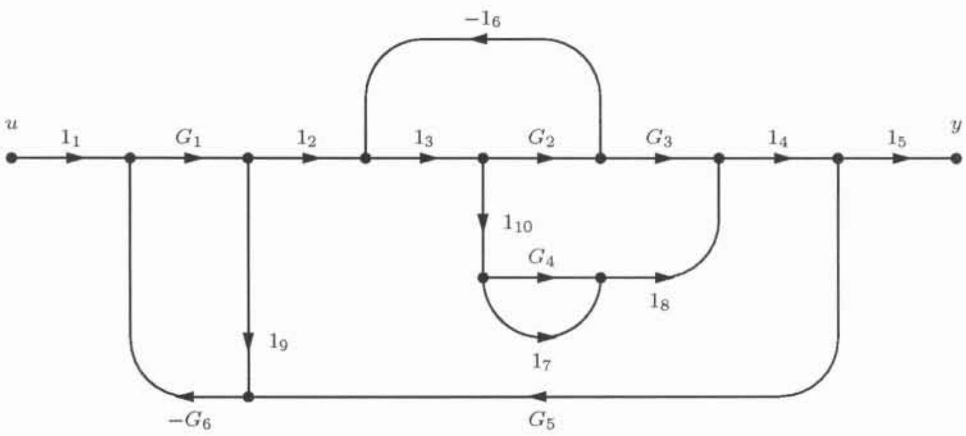








If we choose to use Mason's formula, we need to draw the signal flow graph of the block diagram.



In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The forward path gains are

$$F_1 = 1_1G_11_21_3G_2G_31_41_5 = G_1G_2G_3,$$

$$F_2 = 1_1G_11_21_31_{10}G_41_81_41_5 = G_1G_4,$$

$$F_3 = 1_1G_11_21_31_{10}1_71_81_41_5 = G_1.$$

The loop gains are

$$
\begin{aligned}
L_1 &= G_1 1_9(-G_6) = -G_1G_6,\\
L_2 &= 1_3 G_2(-1_6) = -G_2,\\
L_3 &= G_1 1_2 1_3 G_2 G_3 1_4 G_5(-G_6) = -G_1G_2G_3G_5G_6,\\
L_4 &= G_1 1_2 1_3 1_{10} G_4 1_8 1_4 G_5(-G_6) = -G_1G_4G_5G_6,\\
L_5 &= G_1 1_2 1_3 1_{10} 1_7 1_8 1_4 G_5(-G_6) = -G_1G_5G_6.
\end{aligned}
$$

From the forward path and the loop gains, we determine the touching loops and the forward paths.

Touching Loops

	L_1	L_2	L_3	L_4	L_5
L_1	✔	✘	✔	✔	✔
L_2		✔	✔	✔	✔
L_3			✔	✔	✔
L_4				✔	✔
L_5					✔

Loops on Forward Paths

	L_1	L_2	L_3	L_4	L_5
F_1	✔	✔	✔	✔	✔
F_2	✔	✔	✔	✔	✔
F_3	✔	✔	✔	✔	✔

Therefore,

$$
\begin{aligned}
\Delta &= 1 - \big(L_1 + L_2 + L_3 + L_4 + L_5\big) + \big(L_1L_2\big)\\
&= 1 - \big((-G_1G_6) + (-G_2) + (-G_1G_2G_3G_5G_6) + (-G_1G_4G_5G_6) + (-G_1G_5G_6)\big) + \big((-G_1G_6)(-G_2\\
&= 1 + G_1G_6 + G_2 + G_1G_2G_3G_5G_6 + G_1G_4G_5G_6 + G_1G_5G_6 + G_1G_2G_6,
\end{aligned}
$$

and

$$\Delta_i = \Delta\big|_{L_1=L_2=L_3=L_4=L_5=0} = 1$$

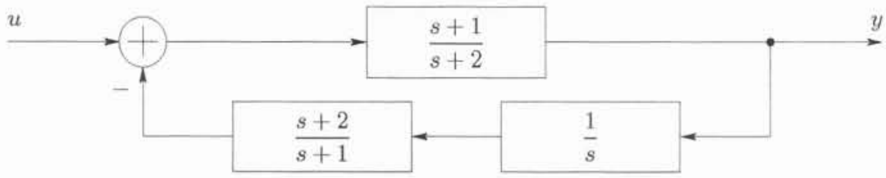
for $i = 1, \ldots, 5$. So,

$$\frac{Y(s)}{U(s)} = \frac{1}{\Delta}\sum_{i=1}^{3} F_i\Delta_i = \frac{(G_1G_2G_3)(1) + (G_1G_4)(1) + (G_1)(1)}{1 + G_1G_6 + G_2 + G_1G_2G_3G_5G_6 + G_1G_4G_5G_6 + G_1G_5G_6 + G_1G_2G_6},$$

or

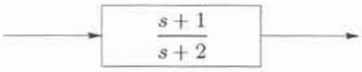
$$\frac{Y(s)}{U(s)} = \frac{G_1G_2G_3 + G_1G_4 + G_1}{1 + G_1G_6 + G_2 + G_1G_2G_3G_5G_6 + G_1G_4G_5G_6 + G_1G_5G_6 + G_1G_2G_6}.$$

3. The block diagram of a control system is given below.

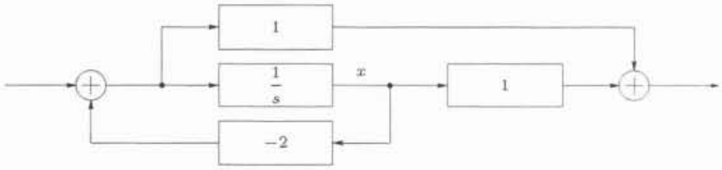


Obtain a state-space representation of the system describing all the internal dynamics.

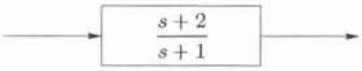
Solution: In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.



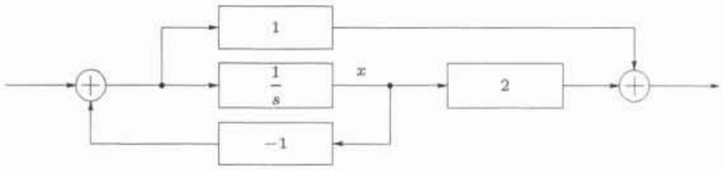
(a) The feedforward gain block.



(b) Controller realization form.

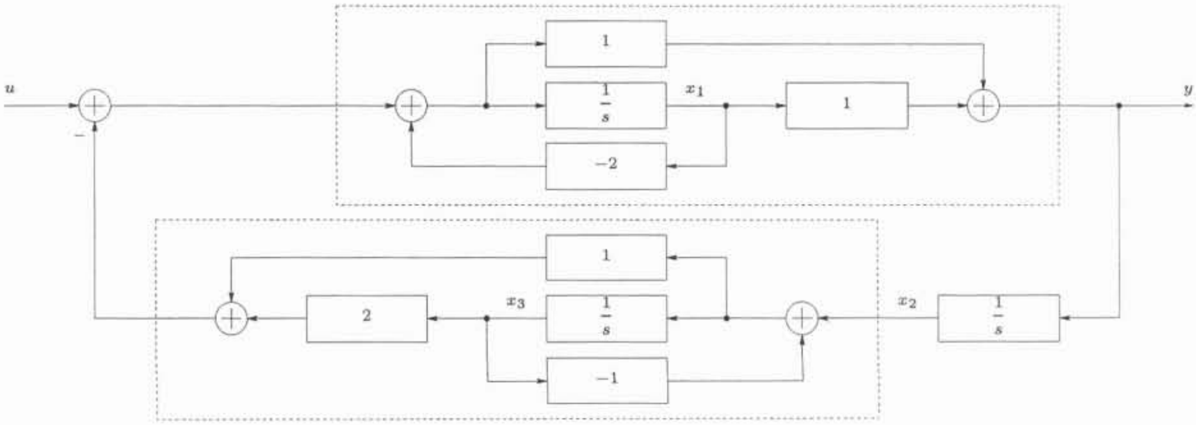


(a) The second feedback gain block.



(b) Controller realization form.

The connected and "expanded" block diagram is shown below.



After assigning the state variables as shown in the figure, we obtain

$$\dot{x}_1 = -2x_1 + \big(u - (2x_3 + \dot{x}_3)\big),$$
$$\dot{x}_2 = x_1 + \dot{x}_1,$$
$$\dot{x}_3 = x_2 - x_3,$$

and

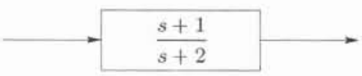
$$y = x_1 + \dot{x}_1.$$

After substituting $\dot{x}_3$ and then $\dot{x}_1$ into the right-hand side of the equations and writing them in matrix form, we obtain the state-space representation

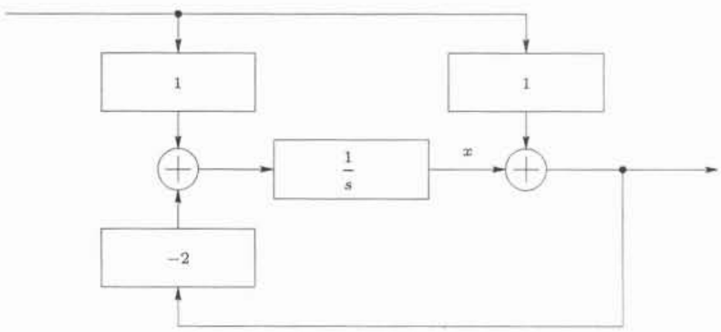
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -2 & -1 & -1 \\ -1 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} u(t).$$

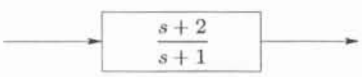
If we use the observer realization form for each of the blocks, then we obtain a different state-space representation.



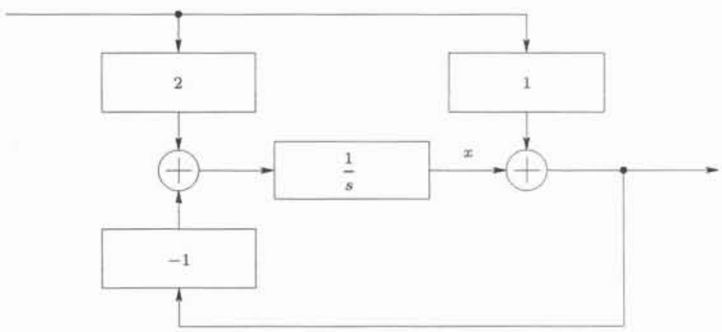
(a) The feedforward gain block.



(b) Observer realization form.

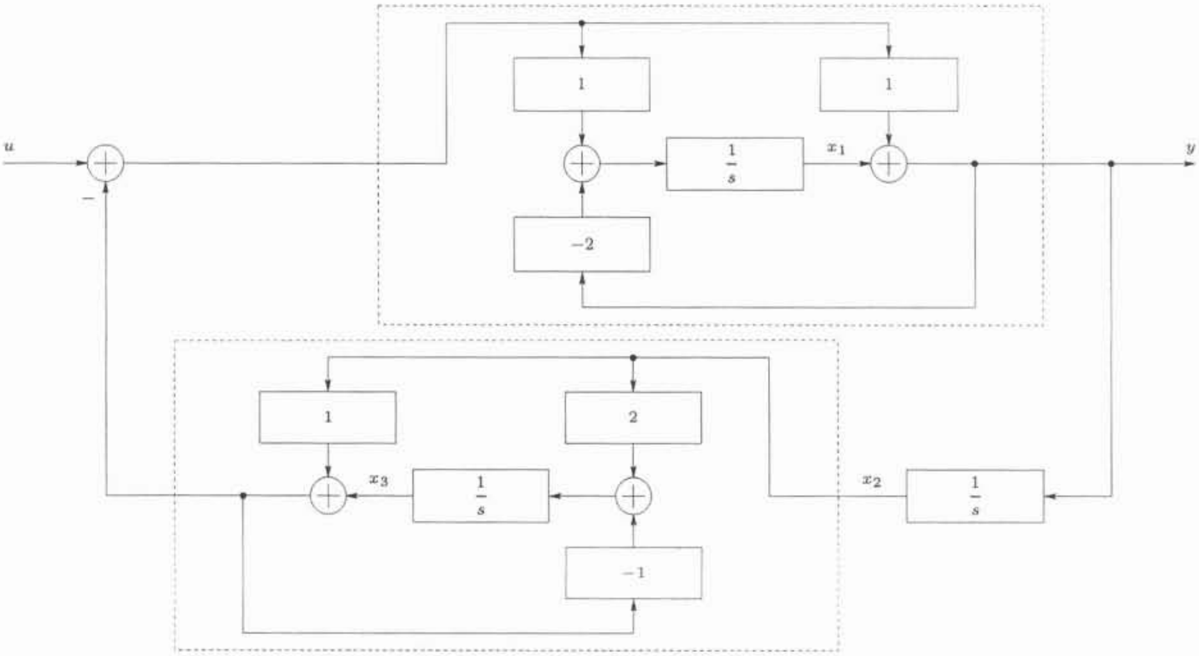


(a) The second feedback gain block.



(b) Observer realization form.

The connected and "expanded" block diagram for this case is shown below.



Similarly, we obtain

$$\begin{aligned}\dot{x}_1 &= u - (x_2 + x_3) - 2y,\\ \dot{x}_2 &= y,\\ \dot{x}_3 &= 2x_2 + (-1)(x_2 + x_3),\end{aligned}$$

and

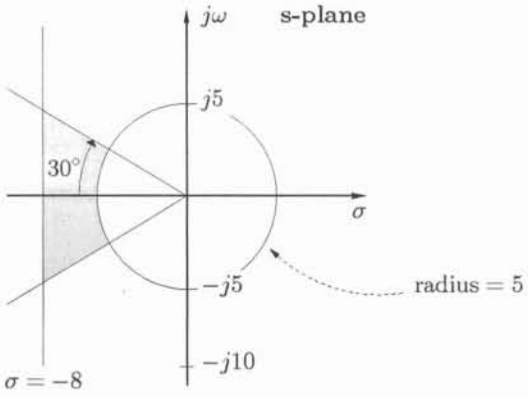
$$y = x_1 + u - (x_2 + x_3).$$

After substituting the y expression into the differential equations and writing them in matrix form, we obtain another state-space representation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} u(t),$$

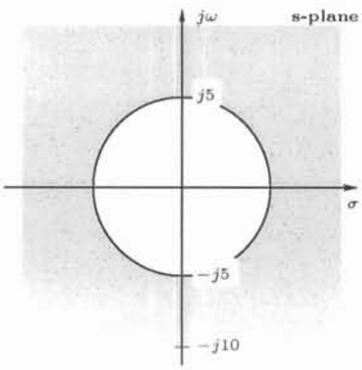
$$y(t) = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} u(t).$$

4. Consider a second-order system described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, such that its poles are located in the shaded region below.



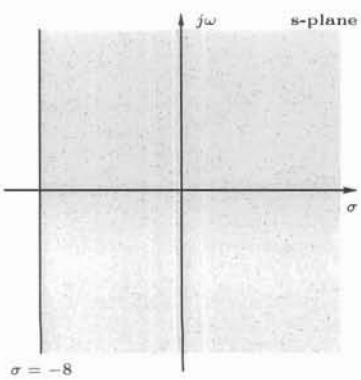
Obtain the necessary inequalities to describe the poles in the shaded region below in terms of only ζ and ω_n.

Solution: To be able to describe the shaded region, we need to separate it into unions or intersections of simpler regions.



The equi-distance points from the origin designate constant value for ω_n. As a result, the shown shaded area is represented by

$$\omega_n \geq 5.$$

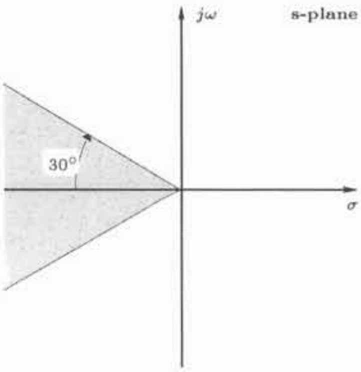


A vertical straight line designates a constant value for the real part of the poles. Since the real part of the complex poles are at $-\zeta\omega_n$, the shown shaded area is represented by

$$-\zeta\omega_n \geq -8,$$

or

$$\zeta\omega_n \leq 8.$$



A straight line originating from the origin designates a constant ζ value, where $\cos^{-1}(\zeta)$ is the acute angle between the line and the negative real axis. So for the shaded area shown, we have

$$0 < \cos^{-1}(\zeta) \leq 30°,$$

or

$$1 > \zeta \geq \cos(30°),$$

since $\cos(\theta)$ is a monotonically decreasing function for $0 < \theta < 180°$. So, we have

$$\frac{\sqrt{3}}{2} \leq \zeta < 1$$

for the complex poles.

The shaded area given in the problem is the intersection of the individual shaded areas.

$$\sqrt{3}/2 \leq \zeta < 1,$$
$$\omega_n \geq 5,$$
$$\zeta\omega_n \leq 8,$$

when the poles have non-zero imaginary parts.

When the poles are real or when $\zeta \geq 1$, they're on the portion of the real axis, such that $-8 \leq \sigma \leq -5$. Since the poles are located at

$$s = -\zeta\omega_n \pm \left(\sqrt{\zeta^2 - 1}\right)\omega_n,$$

the smaller one needs to be greater than -8

$$-\zeta\omega_n - \left(\sqrt{\zeta^2 - 1}\right)\omega_n \geq -8,$$

and the larger one needs to be less than -5,

$$-\zeta\omega_n + \left(\sqrt{\zeta^2 - 1}\right)\omega_n \leq -5,$$

or

$$\zeta \geq 1,$$
$$\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n \geq 5,$$
$$\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n \leq 8,$$

when the poles are real.