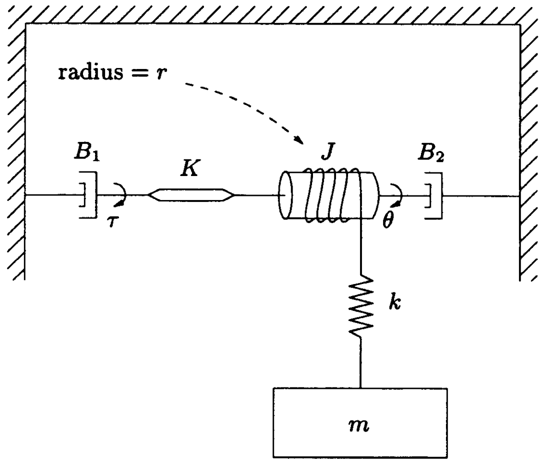
EE 231 **Exam#1** **Mar. 08, 2005**

75 minutes

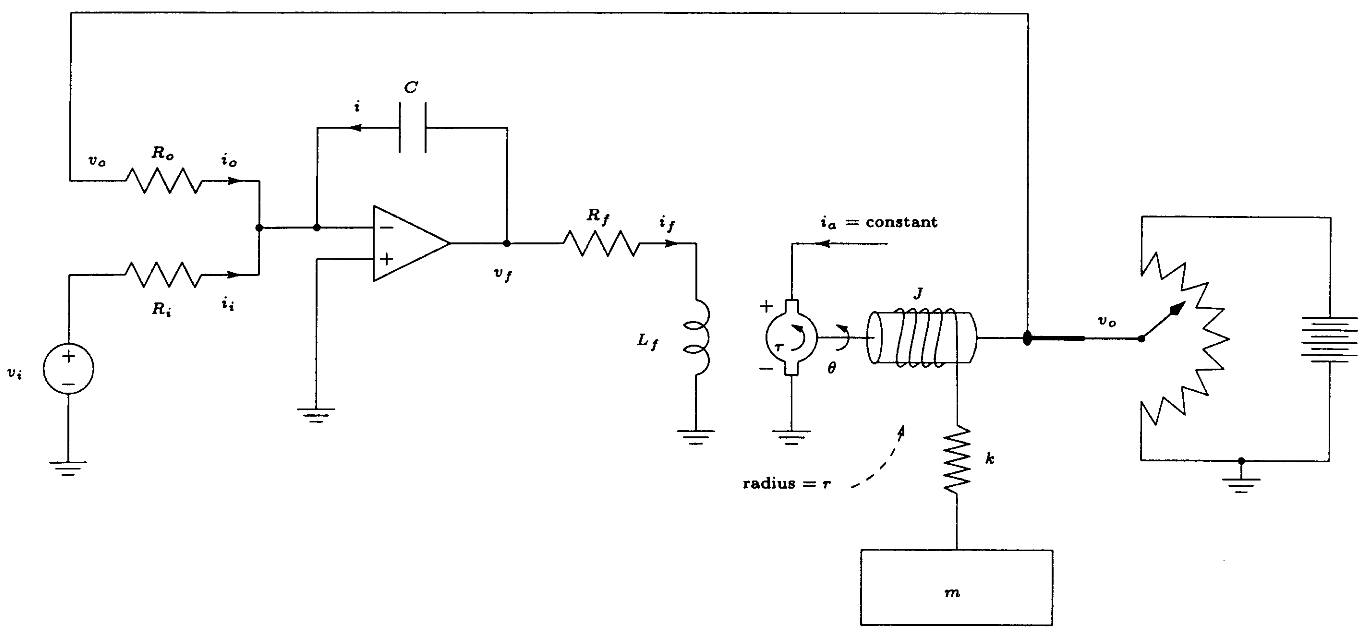
1. Consider the mechanical system shown below.



(a) Obtain the transfer function of the system assuming that the external variable τ is the input, and the angle θ is the output. Assume that the spring with the spring constant k is stretched due to the gravity prior to the application of the torque τ. (15pts)

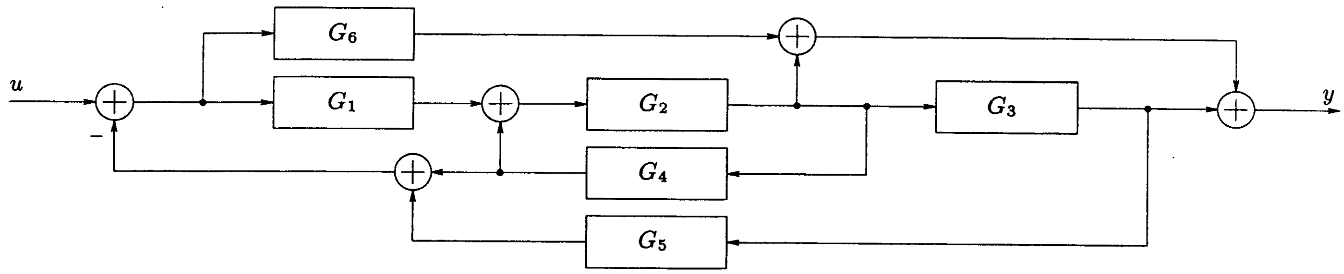
(b) Obtain *either* the torque/force-voltage *or* the torque/force-current analog of the system. (10pts)

2. The angular position of the shaft of a motor is controlled by the system shown below.



The angular position of the motor shaft is detected by a variable resistor which provides a voltage v_o proportional to the angle, such that $v_o = K_o\theta$. Draw the most detailed block diagram of the system, where v_i is the input, and θ is the output. Show all the variables v_i, i_i, v_o, i_o, i, v_f, i_f, τ, and θ as well as the displacement(s) associated with the mass-spring components on the block diagram. (25pts)

3. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly. (25pts)



4. A control system is represented by

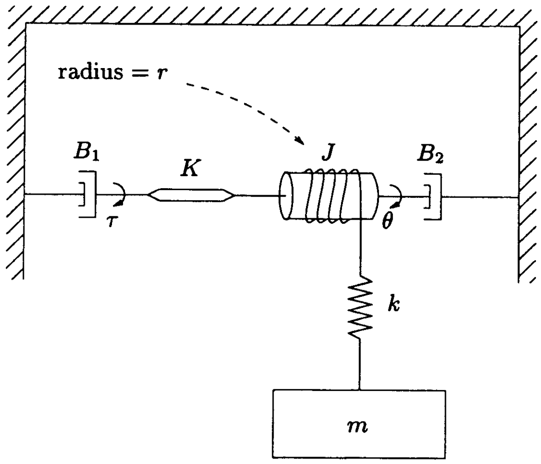
$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively. Determine $y(t)$ for $t \geq 0$; when $\boldsymbol{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, and $u(t) = 1$ for $t \geq 0$. (25pts)

EE 231 **Exam#1 Solutions** Mar. 08, 2005

1. Consider the mechanical system shown below.

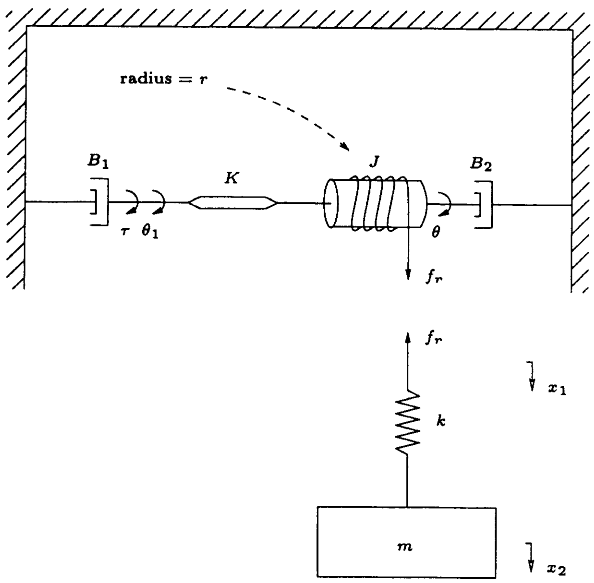


(a) Obtain the transfer function of the system assuming that the external variable τ is the input, and the angle θ is the output. Assume that the spring with the spring constant k is stretched due to the gravity prior to the application of the torque τ.

Solution:

First, we separate the rotational system from the translational system by representing the internal force.

Second, we identify the linearly independent rotations and displacements in the mechanical systems and mark them.



Third, we write the differential equations describing the the rotational motion from the mechanical system.

$$0 = \tau - B_1\dot{\theta}_1 - K(\theta_1 - \theta)$$

$$J\ddot{\theta} = rf_r - B_2\dot{\theta} - K(\theta - \theta_1).$$

Then, we write the differential equations describing the the translational motion.

$$0 = -f_r - k(x_1 - x_2)$$

$$m\ddot{x}_2 = -k(x_2 - x_1).$$

From the connection, we also know that

$$x_1 = r\theta.$$

Next, we obtain the transfer function by taking the Laplace transforms of the above equations under zero initial conditions. After some manipulations, we get

$$(B_1 s + K)\Theta_1(s) - K\Theta(s) = T(s), \quad (1.1)$$

$$(Js^2 + B_2 s + K)\Theta(s) - K\Theta_1(s) = rF_r(s), \quad (1.2)$$

and

$$kX_1(s) - kX_2(s) = -F_r(s), \quad (1.3)$$

$$(ms^2 + k)X_2(s) - kX_1(s) = 0 \quad (1.4)$$

with

$$X_1(s) = r\Theta(s), \quad (1.5)$$

where Θ_1, Θ, X_1, X_2, F_r, and T are the Laplace transforms of θ_1, θ, x_1, x_2, f_r, and τ, respectively. From Equations (1.3) and (1.4), we get

$$X_2(s) = -\frac{1}{ms^2}F_r(s),$$

and

$$X_1(s) = -\left(\frac{ms^2 + k}{mks^2}\right)F_r(s).$$

Solving for F_r in terms of Θ from the last equation along with Equation (1.5), we get

$$F_r(s) = -\left(\frac{mks^2}{ms^2 + k}\right)X_1(s) = -\left(\frac{mks^2}{ms^2 + k}\right)r\Theta(s).$$

We solve for Θ_1 by using Equation (1.1) to obtain

$$\Theta_1(s) = \frac{K}{B_1 s + K}\Theta(s) + \frac{1}{B_1 s + K}T(s).$$

Finally, substituting Θ_1 and F_r in Equation (1.2), we get

$$(Js^2 + B_2 s + K)\Theta(s) - K\left(\frac{K}{B_1 s + K}\Theta(s) + \frac{1}{B_1 s + K}T(s)\right) = -\left(\frac{mkr^2 s^2}{ms^2 + k}\right)\Theta(s),$$

or

$$\left((Js^2 + B_2 s + K) - \frac{K^2}{B_1 s + K} + \frac{mkr^2 s^2}{ms^2 + k}\right)\Theta(s) = \frac{K}{B_1 s + K}T(s).$$

Therefore,

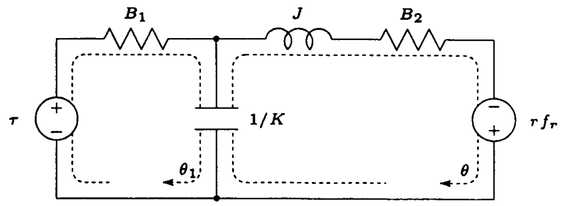
$$\frac{\Theta(s)}{T(s)} = \frac{K(ms^2 + k)}{\left((Js^2 + B_2 s + K)(B_1 s + K) - K^2\right)(ms^2 + k) + mkr^2 s^2 (B_1 s + K)},$$

where Θ and T are the Laplace transforms of θ and τ, respectively.

(b) Obtain *either* the torque/force-voltage *or* the torque/force-current analog of the system.

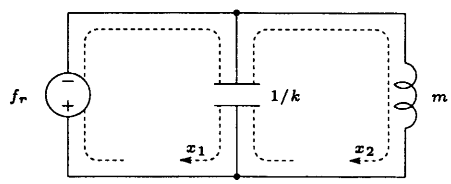
Solution: For the torque-voltage analog of a rotational mechanical system, there will be a loop charge associated with each rotational variable, and an input torque will be associated with a voltage source. The stiffness constant, the rotational damping-constant, and the inertia will be associated with the reciprocal of capacitance, the resistance, and the inductance, respectively. The elements between two rotational variables of the mechanical system will be between the corresponding loop variables of the torque-voltage analog. The elements that are connected to fixed frames and the elements that are always measured with respect to a fixed frame, such as the inertia and the external torque, will be on the non-common portions of the loops.

The next figure shows the torque-voltage analog of the rotational system, where the loops are identified by the angles.

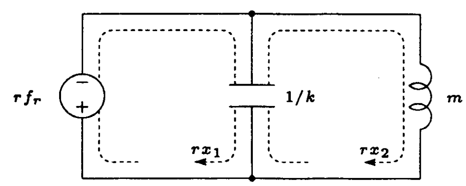


For the force-voltage analog of a translational mechanical system, there will be a loop charge associated with each displacement variable (or a loop current associated with each velocity variable), and an input force will be associated with a voltage source. The spring constant, the damping constant, and the mass will be associated with the reciprocal of capacitance, the resistance, and the inductance, respectively. The elements between two displacement variables of the mechanical system will be between the corresponding loop variables of the force-voltage analog. The elements that are connected to fixed frames and the elements that are always measured with respect to a fixed frame, such as the mass and the external force, will be on the non-common portions of the loops.

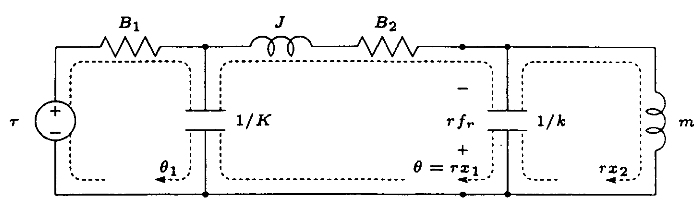
The next figure shows the force-voltage analog of the translational system, where the loops are identified by the displacements.



In order to merge the rotational and the translational analogs, we need to match the voltage supply representing the internal force variable f_r. To preserve the angle θ representation, we scale the voltage-supply value of the force-voltage analog as shown in the next figure.

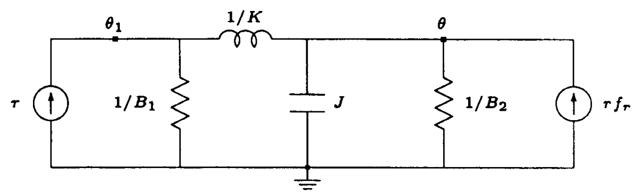


The next figure shows the torque/force-voltage analog of the mechanical system, where the loops are identified by the angles and the displacements.



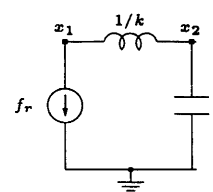
For the torque-current analog of a rotational mechanical system, there will be a node flux associated with each rotational variable, and an input torque will be associated with a current source. The stiffness constant, the rotational damping-constant, and the inertia will be associated with the reciprocal of inductance, the conductance, and the capacitance, respectively. The elements between two rotational variables of the mechanical system will be between the corresponding node variables of the torque-current analog. The elements that are connected to fixed frames and the elements that are always measured with respect to a fixed frame, such as the inertia and the external torque, will be connected to the ground.

The next figure shows the torque-current analog of the rotational system, where the nodes are identified by the angles.

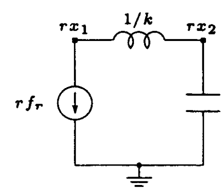


For the force-current analog of a translational mechanical system, there will be a node flux associated with each displacement variable (or a node voltage associated with each velocity variable), and an input force will be associated with a current source. The spring constant, the damping constant, and the mass will be associated with the reciprocal of inductance, the conductance, and the capacitance, respectively. The elements between two displacement variables of the mechanical system will be between the corresponding node variables of the force-current analog. The elements that are connected to fixed frames and the elements that are always measured with respect to a fixed frame, such as the mass and the external force, will be connected to the ground.

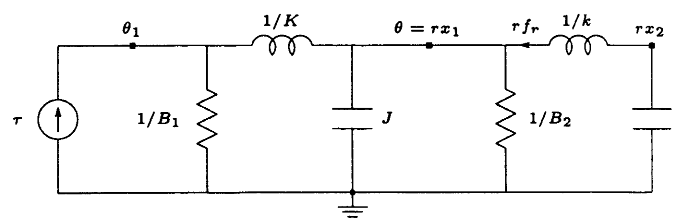
The next figure shows the force-current analog of the translational system, where the nodes are identified by the displacements.



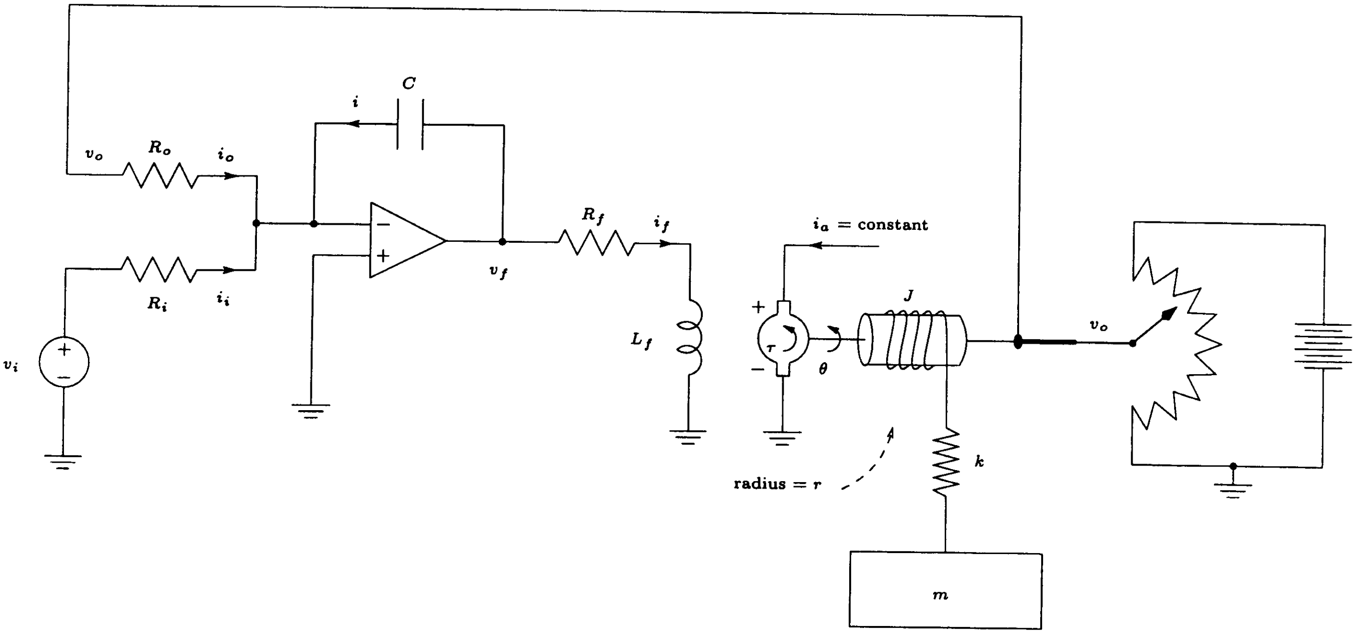
In order to merge the rotational and the translational analogs, we need to match the current supply representing the internal force variable f_r. To preserve the angle θ representation, we scale the current-supply value of the force-current analog as shown in the next figure.



The next figure shows the torque/force-current analog of the mechanical system, where the nodes are identified by the angles and the displacements.

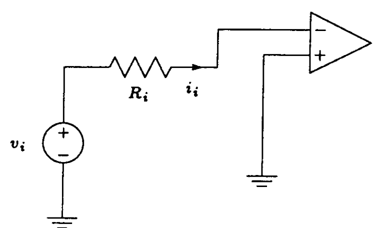


2. The angular position of the shaft of a motor is controlled by the system shown below.



The angular position of the motor shaft is detected by a variable resistor which provides a voltage v_o proportional to the angle, such that $v_o = K_o\theta$. Draw the most detailed block diagram of the system, where v_i is the input, and θ is the output. Show all the variables v_i, i_i, v_o, i_o, i, v_f, i_f, τ, and θ as well as the displacement(s) associated with the mass-spring components on the block diagram.

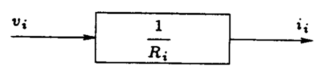
Solution: To determine the block diagram of the system, we first separate it into simpler components.

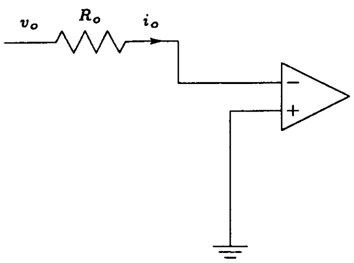


Since the input variable is v_i, we write i_i in terms v_i, such that

$$I_i(s) = \frac{1}{R_i}V_i(s),$$

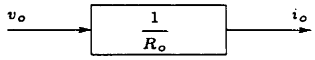
since the operational amplifier is assumed to be ideal.

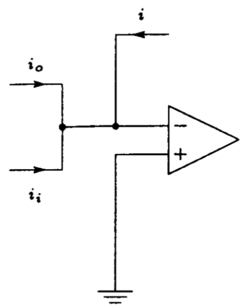




Similarly, we have

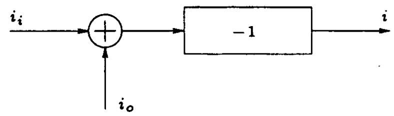
$$I_o(s) = \frac{1}{R_o} V_o(s).$$

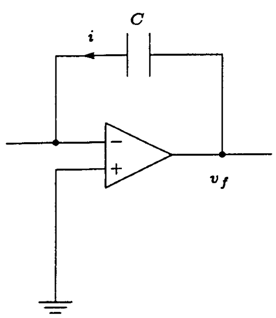




For an ideal operational amplifier,

$$i(t) = -\big(i_i(t) + i_o(t)\big).$$



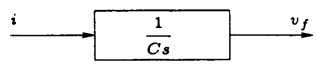


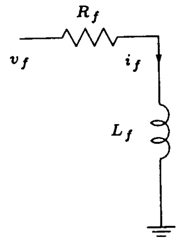
Again for an ideal operational amplifier,

$$v_f(t) = \frac{1}{C}\int^t i(\tau)\,\mathrm{d}\tau,$$

or

$$V_f(s) = \frac{1}{Cs} I(s).$$



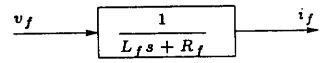


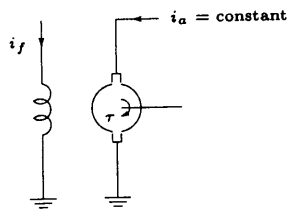
The field current of the motor can be obtained from the Kirchhoff's Voltage Law, where

$$L_f\frac{\mathrm{d}i_f(t)}{\mathrm{d}t}+R_f i_f(t)=v_f(t),$$

or

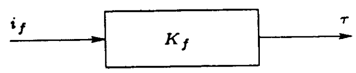
$$I_f(s)=\frac{1}{L_f s+R_f}V_f(s).$$

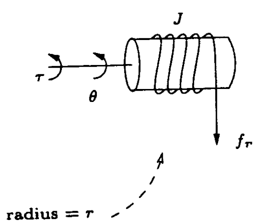




From the field controlled motor,

$$\tau(t)=K_f i_f(t).$$



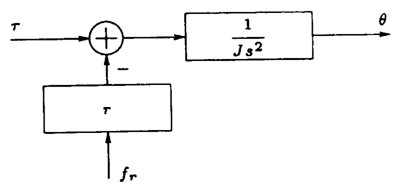


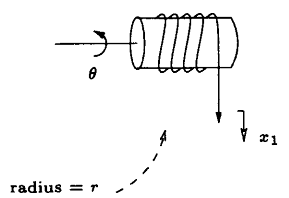
The torque equation for θ is

$$J\frac{\mathrm{d}^2\theta_m(t)}{\mathrm{d}t^2}=\tau(t)-rf_r(t),$$

where f_r is the internal tension of the rope. So,

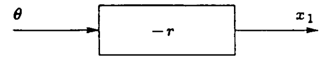
$$\Theta(s)=\frac{1}{Js^2}\big(T(s)-rF_r(s)\big).$$

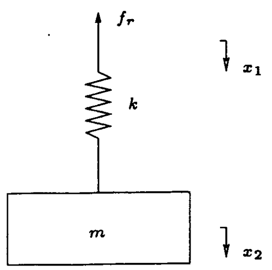




The disc with the inertia J changes the rotational motion to translational motion, where

$$x_1(t) = -r\theta(t).$$



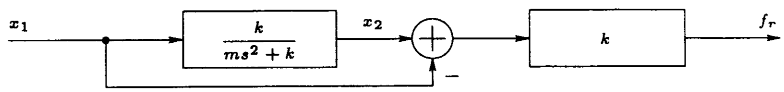


The differential equations describing the the translational motion are

$$0 = -f_r - k(x_1 - x_2),$$
$$m\ddot{x}_2 = -k(x_2 - x_1).$$

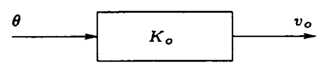
So,

$$F_r(s) = k\big(X_2(s) - X_1(s)\big),$$
$$X_2(s) = \frac{k}{ms^2 + k}X_1(s).$$

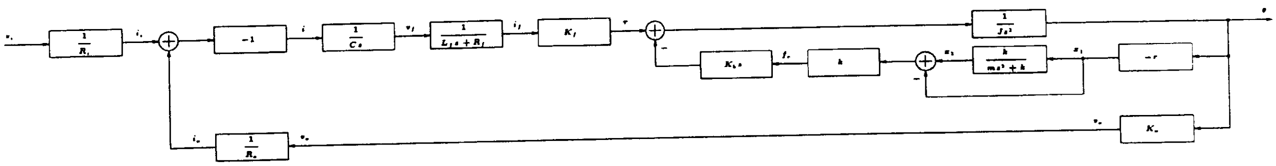


And, finally the given relationship

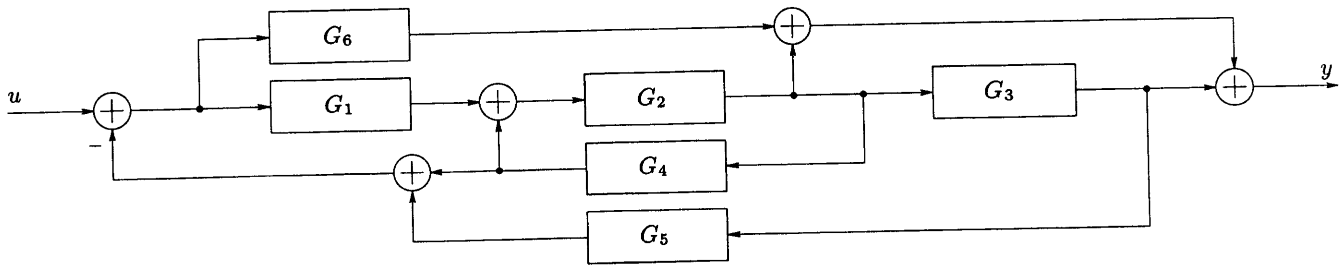
$$v_o(t) = K_o\theta(t).$$



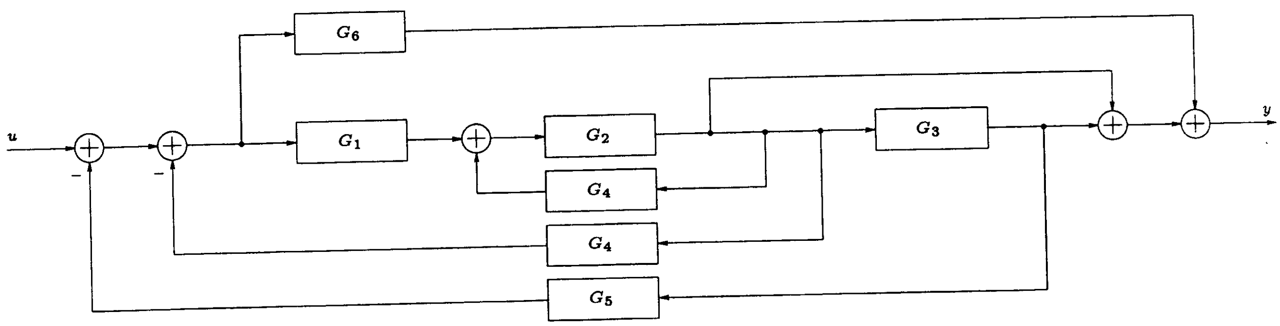
When we connect all the individual blocks together, we get the following block diagram.

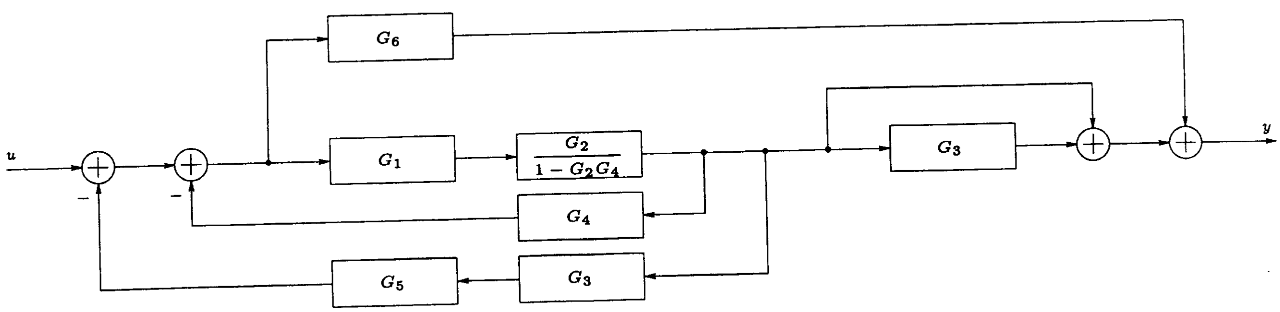


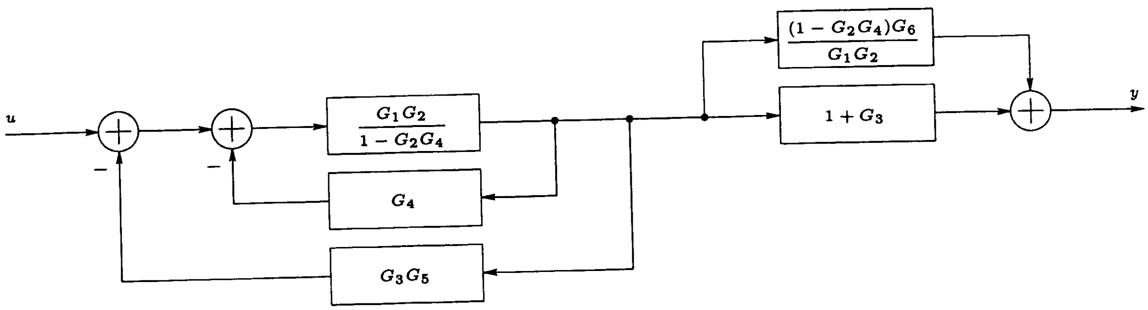
3. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly.

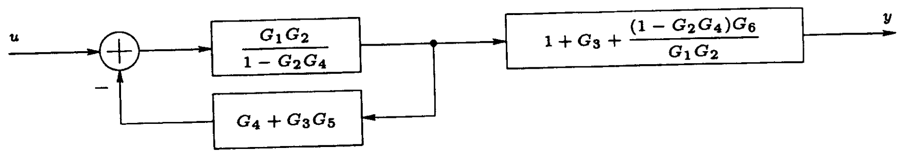


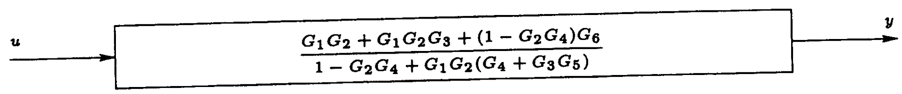
Solution: If we choose to use the block-diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.



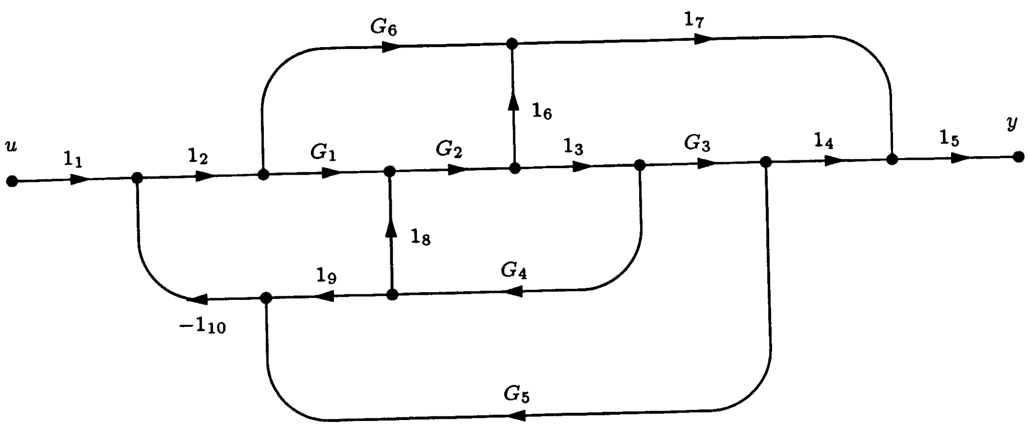








If we choose to use Mason's formula, we need to draw the signal flow graph of the block diagram.



In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The forward path gains are

$$F_1 = 1_1 1_2 G_1 G_2 1_3 G_3 1_4 1_5 = G_1G_2G_3,$$

$$F_2 = 1_1 1_2 G_6 1_7 1_5 = G_6,$$

$$F_3 = 1_1 1_2 G_1 G_2 1_6 1_7 1_5 = G_1G_2.$$

The loop gains are

$$L_1 = 1_1 1_2 G_1 G_2 1_3 G_3 G_5(-1_{10}) = -G_1G_2G_3G_5,$$
$$L_2 = 1_1 1_2 G_1 G_2 1_3 G_4 1_9(-1_{10}) = -G_1G_2G_4,$$
$$L_3 = G_2 1_3 G_4 1_8 = G_2G_4.$$

From the forward path and the loop gains, we determine the touching loops and the forward paths.

Touching Loops

	L_1	L_2	L_3
L_1	✔	✔	✔
L_2		✔	✔
L_3			✔

Loops on Forward Paths

	L_1	L_2	L_3
F_1	✔	✔	✔
F_2	✔	✔	✘
F_3	✔	✔	✔

Therefore,

$$\begin{aligned}\Delta &= 1 - (L_1 + L_2 + L_3)\\ &= 1 - ((-G_1G_2G_3G_5) + (-G_1G_2G_4) + (G_2G_4))\\ &= 1 + G_1G_2G_3G_5 + G_1G_2G_4 - G_2G_4,\end{aligned}$$

and

$$\Delta_1 = \Delta|_{L_1=L_2=L_3=0} = 1,$$
$$\Delta_2 = \Delta|_{L_1=L_2=0} = 1 - L_3 = 1 - G_2G_4,$$
$$\Delta_3 = \Delta|_{L_1=L_2=L_3=0} = 1.$$

So,

$$\frac{Y(s)}{U(s)} = \frac{1}{\Delta}\sum_{i=1}^{3} F_i\Delta_i = \frac{(G_1G_2G_3)(1) + (G_6)(1 - G_2G_4) + (G_1G_2)(1)}{1 + G_1G_2G_3G_5 + G_1G_2G_4 - G_2G_4},$$

or

$$\frac{Y(s)}{U(s)} = \frac{G_1G_2G_3 + G_6(1 - G_2G_4) + G_1G_2}{1 + G_1G_2G_3G_5 + G_1G_2G_4 - G_2G_4}.$$

4. A control system is represented by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{x}(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively. Determine $y(t)$ for $t \geq 0$; when $\boldsymbol{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, and $u(t) = 1$ for $t \geq 0$.

Solution: The general solution to the state-space representation of a system described by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t)$$

is obtained from

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)}B\boldsymbol{u}(\tau)\,\mathrm{d}\tau,$$

where

$$e^{At} = \mathcal{L}_s^{-1}\left[(sI - A)^{-1}\right](t).$$

Here, I is the appropriately dimensioned identity matrix. In our case,

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \end{bmatrix},$$

$\boldsymbol{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, and $u(t) = 1$ for $t \geq 0$. As a result, the initial-condition term in the solution of $\boldsymbol{x}$ is identically zero. So,

$$y(t) = C\int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau + Du(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\int_0^t \mathcal{L}_s^{-1}\left[(sI-A)^{-1}\right](t-\tau)\begin{bmatrix} 1 \\ 0 \end{bmatrix}(1)\,\mathrm{d}\tau$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix}\int_0^t \mathcal{L}_s^{-1}\left[\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}\right)^{-1}\right](t-\tau)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathrm{d}\tau$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix}\int_0^t \mathcal{L}_s^{-1}\left[\left(\begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix}\right)^{-1}\right](t-\tau)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathrm{d}\tau$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix}\int_0^t \mathcal{L}_s^{-1}\left[\frac{1}{(s+1)(s+2)}\begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix}\right](t-\tau)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathrm{d}\tau$$

$$= \int_0^t \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \mathcal{L}_s^{-1}\left[\frac{1}{s+1}\right](t-\tau) & \mathcal{L}_s^{-1}\left[\frac{1}{(s+1)(s+2)}\right](t-\tau) \\ 0 & \mathcal{L}_s^{-1}\left[\frac{1}{s+2}\right](t-\tau) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathrm{d}\tau$$

$$= \int_0^t \mathcal{L}_s^{-1}\left[\frac{1}{s+1}\right](t-\tau)\,\mathrm{d}\tau = \int_0^t e^{-(t-\tau)}\,\mathrm{d}\tau = \left(e^{-(t-\tau)}\right)_{\tau=0}^{\tau=t}.$$

Or,

$$y(t) = 1 - e^{-t} \text{ for } t \geq 0.$$