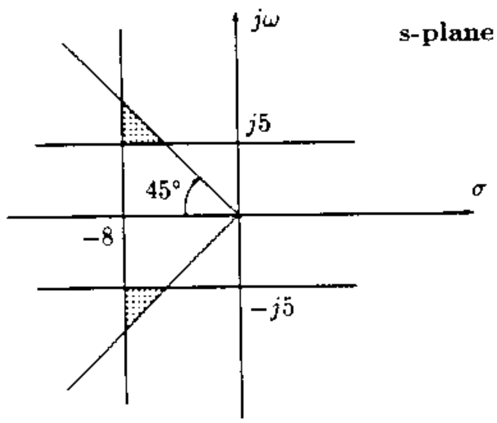
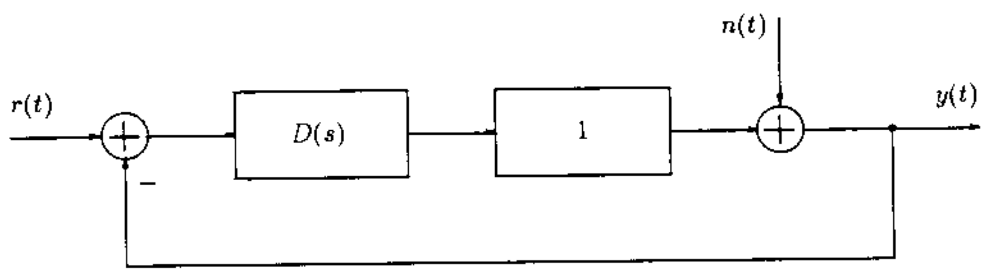
EE 231 | **Exam#2** 75 minutes | **Apr. 14, 1992**

1. Obtain the necessary inequalities to describe the poles in the shaded region in terms of only ζ and ω_n of a second-order system described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$. (15pts)



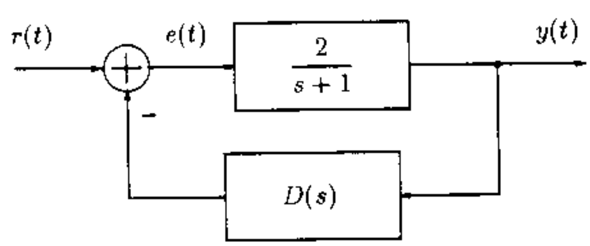
2. A feedback control system is given below.



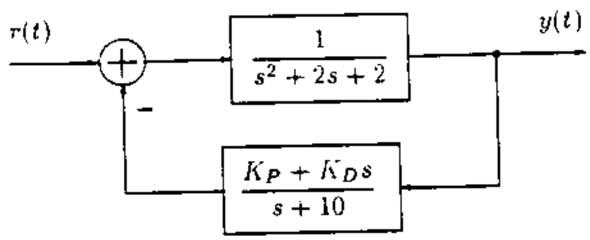
Design the simplest controller $D(s)$, so that the output, $y(t)$, follows a sinusoidal input, $r(t)$, with a frequency of 1 rad/sec, and it rejects a sinusoidal noise, $n(t)$, with a frequency of 4 rad/sec. (30pts)

3. Design a $D(s)$, such that the following system satisfies these conditions.
   (a) The 5% settling time is less than 1 sec.
   (b) The rise time is less than 0.6 sec.
   (c) A zero steady-state error $e(t)$ is obtained for a step reference input.

(30pts)



4. A modified PD controller is to be designed for the following system. Determine the range of controller constants K_P and K_D for an asymptotically stable system. (25pts)

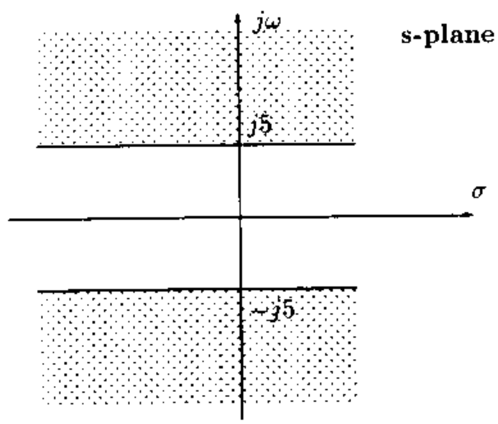


EE 231 **Exam#2** **Apr. 21, 1992**

Solutions

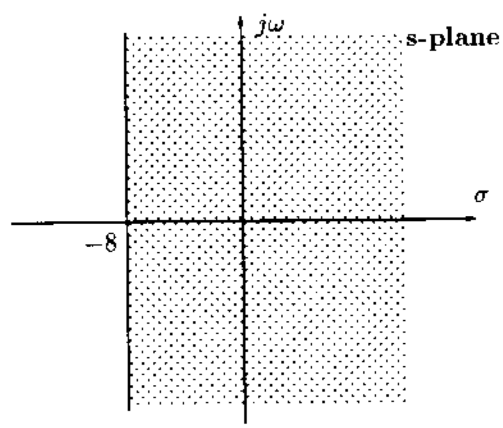
1. The shaded region can be separated into unions and intersections of simpler regions. The region above the horizontal line $j\omega = j5$ shown below is described by

$$\omega_d \geq 5$$
$$\sqrt{1-\zeta^2}\,\omega_n \geq 5.$$



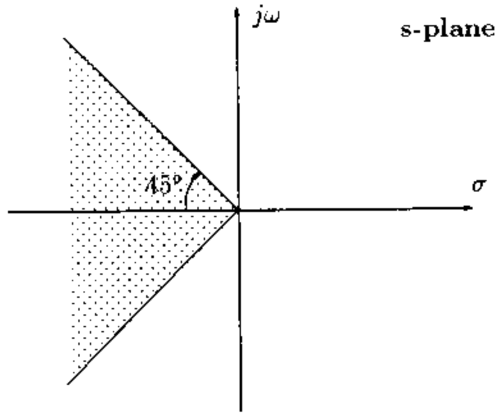
The region to the right of the vertical line $\sigma = -8$ shown below is described by

$$-\zeta\omega_n \geq -8$$
$$\zeta\omega_n \leq 8.$$



The region under the 45° angle line shown below is described by

$$\cos^{-1}\zeta \leq 45^\circ$$
$$\zeta \geq \cos 45^\circ$$
$$\zeta \geq \frac{\sqrt{2}}{2}.$$



Therefore, the given shaded region which is the intersection of all the shaded regions above, is described by all ζ and ω_n which satisfy

$$\sqrt{1-\zeta^2}\,\omega_n \geq 5 \quad \text{and} \quad \zeta\omega_n \leq 8 \quad \text{and} \quad \zeta \geq \frac{\sqrt{2}}{2}.$$

2. In order for the system to follow any non-asymptotically stable poles of the reference, its open-loop gain, i.e., $D(s)$ in this case, has to match the poles of the reference. As a result, to follow a sinusoidal input with frequency 1 rad/sec, we need poles at $s = \pm j1$. In other words,

$$D(s) = \frac{z(s)}{(s^2+1^2)p_1(s)},$$

where $z(s)$ and $p_1(s)$ are (yet) unknown polynomials in s. Moreover to reject disturbance, the open-loop gain also needs to match the non-asymptotically stable poles of the disturbance. In other words,

$$D(s) = \frac{z(s)}{(s^2+4^2)p_2(s)},$$

where $p_2(s)$ is another unknown polynomial in s.

Comparing the two forms of $D(s)$, we realize that the simplest open-loop gain, i.e., the simplest controller, should be

$$D(s) = \frac{z(s)}{(s^2+1)(s^2+16)},$$

which satisfies both forms of $D(s)$ by picking $p_1(s) = s^2+16$, and $p_2(s) = s^2+1$. We now need to select $z(s)$, such that the closed-loop system is stable. Here, we can use Routh-Hurwitz criterion, or solve the characteristic equation of the system. The characteristic equation with the $D(s)$ above, becomes

$$1 + D(s) = 0,$$

$$1 + \frac{z(s)}{(s^2+1)(s^2+16)} = 0,$$

$$(s^2+1)(s^2+16) + z(s) = 0,$$

or

$$s^4 + 17s^2 + 16 + z(s) = 0.$$

To make the system stable, we at least need to supply the missing terms in the characteristic polynomial, i.e.,

$$z(s) = as^3 + bs,$$

for some a and b. The characteristic polynomial is then $s^4 + as^3 + 17s^2 + bs + 16$, and Routh-Hurwitz table becomes

s^4	1	17	16
s^3	a	b	
s^2	$\frac{17a-b}{a}$	16	
s	$\frac{(17a-b)b/a - 16a}{(17a-b)/a}$		
1	16		

For stability,

1. $a > 0$,

2. $\frac{17a-b}{a} > 0$,

   $17a - b > 0$, since $a > 0$,

   $b < 17a$,

3. $\frac{(17a-b)b/a - 16a}{(17a-b)/a} > 0$,

   $\frac{(17a-b)b - 16a^2}{(17a-b)} > 0$, since $a > 0$,

   $(17a-b)b - 16a^2 > 0$, since $17a - b > 0$,

   $-(16a^2 - 17ab + b^2) > 0$,

   $-(a-b)(16a-b) > 0$,

   $(a-b)(b-16a) > 0$,

   a. $a - b > 0$ and $b - 16a > 0$,

   $a > b$ and $b > 16a$, impossible, since $a > 0$, and $16a > a$,

   b. $a - b < 0$ and $b - 16a < 0$,

   $a < b$ and $b < 16a$.

Therefore, the stability requirements can be compactly written as

$$0 < a < b < 16a < 17a.$$

One choice for the simplest controller is $a = 1$, and $b = 2$, i.e.,

$$D(s) = \frac{s^3 + 2s}{(s^2+1)(s^2+16)}.$$

3. Here is a list of the given requirements and corresponding system restrictions.

Given Requirements	General System Restrictions	Specific System Restrictions
Zero steady-state error for a step input	System is Type 1.	For this example, $\frac{2D(s)}{s+1} = \frac{z(s)}{sp(s)}$.
Less than 1 sec 5% settling time	$t_{s_{5\%}} \leq 1$.	For a second-order system, $\frac{3}{\zeta\omega_n} \leq 1$.
Less than 0.6 sec rise time	$t_r \leq 0.6$.	For a second-order system, without a zero, $\frac{\pi - \cos^{-1}\zeta}{\omega_d} \leq 0.6$.

From the zero steady-state requirement, we know that $D(s) = K$ will not work, since it will not increase the type of the system to 1. So, we first try the $D(s) = K/s$, an integral controller. The transfer function then becomes

$$\frac{Y(s)}{R(s)} = \frac{2s}{s^2 + s + 2K}.$$

This is a second-order system, and from the representation of a general second-order system, we know that $2\zeta\omega_n = 1$, and $\omega_n^2 = 2K$. But the 5% settling time requirement implies $\zeta\omega_n \geq 3$, or $2\zeta\omega_n \geq 6$. As a result, no value of K will satisfy the settling time requirement.

Next, in order not to make the order of the system more than 2, we try a proportional-integral controller, i.e., $D(s) = K_P + K_I/s$. The transfer function becomes

$$\frac{Y(s)}{R(s)} = \frac{2s}{s^2 + (2K_P + 1)s + 2K_I}.$$

Similarly, from the representation of a general second-order system, we know that $2\zeta\omega_n = 2K_P + 1$, and $\omega_n^2 = 2K_I$. Since the 5% settling time requirement implies $2\zeta\omega_n \geq 6$, we have $2K_P + 1 \geq 6$, or $K_P \geq 5/2$. Let $K_P = 3$, such that the transfer function becomes

$$\frac{Y(s)}{R(s)} = \frac{2s}{s^2 + 7s + 2K_I}.$$

However, since the transfer function has a zero at zero, the rise time condition becomes useless, and any $K_I > 0$ is an acceptable solution.

4. The characteristic equation of the system is

$$1 + \frac{1}{s^2 + 2s + 2}\frac{K_p + K_D s}{s + 10} = 0,$$

$$(s^2 + 2s + 2)(s + 10) + (K_p + K_D s) = 0.$$

or

$$s^3 + 12s^2 + (K_D + 22)s + (K_P + 20) = 0.$$

To determine the range of asymptotical stability, we use Routh-Hurwitz stability criterion.

$$\begin{array}{c|cc} s^3 & 1 & K_D + 22 \\ s^2 & 12 & K_P + 20 \\ s & \dfrac{12(K_D + 22) - (K_P + 20)}{12} & \\ 1 & K_P + 20 & \end{array}$$

For stability,

1. $$\frac{12(K_D + 22) - (K_P + 20)}{12} > 0,$$
   $$12K_D - K_P + 244 > 0,$$
   $$12K_D - K_P > -244 > 0,$$

2. $$K_P + 20 > 0,$$
   $$K_P > -20.$$

Therefore, the stability requirements can be written as

$$12K_D - K_P > -244, \qquad \text{and} \qquad K_P > -20.$$