EE 231 | Exam#2 | May 03, 1999

50 minutes

1. Consider a unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{s^2 - 4s + 13}{(s+1)(s+4)(s+5)}.$$

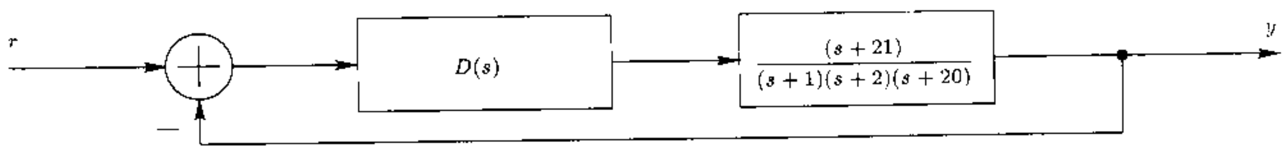
Determine the range of K for the asymptotical stability of the closed-loop system. (20pts)

2. Consider a unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{s^2 + 10s + 50}{s^2(s+10)(s+15)} = K\frac{s^2 + 10s + 50}{s^4 + 25s^3 + 150s^2}.$$

Construct the root-locus diagram. Determine all the important necessary features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, and angle of arrivals and departures. (25pts)

3. Consider the following control system.



(a) Design a first order compensator $D(s) = D_1(s)$, such that the steady state error is zero for a step input without increasing the order of the system. If there exists a freedom of choice in the location of the compensator poles and/or zeros, choose to minimize the settling time. (20pts)

(b) Design another first order compensator $D_2(s)$ cascaded to the previous compensator, i.e. $D(s) = D_1(s)D_2(s)$, such that the desired dominant closed-loop poles are at $s_d = -4 \pm j2$. (20pts)

(c) Design a different compensator $D(s)$ for the original system, such that the non-zero and finite steady state error is decreased by 5 times with minimal effect on the existing closed-loop poles. Assume that the slowest physically realizable stable pole of the compensator is at -0.01. (15pts)

EE 231 — **Exam#2 Solutions** — May 03, 1999

1. Consider a unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{s^2 - 4s + 13}{(s+1)(s+4)(s+5)}.$$

Determine the range of K for the asymptotical stability of the closed-loop system.

Solution: The stability of the closed-loop system can be determined using the Routh-Hurwitz's stability criterion on the characteristic polynomial. From the characteristic equation, $1 + G(s) = 0$,

$$1 + K\frac{s^2 - 4s + 13}{(s+1)(s+4)(s+5)} = 0,$$

$$(s+1)(s+4)(s+5) + K(s^2 - 4s + 13) = 0,$$

or

$$s^3 + (10+K)s^2 + (29-4K)s + (20+13K) = 0.$$

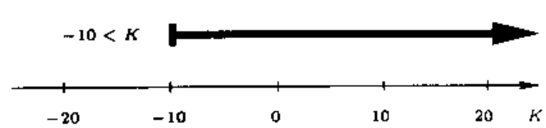
The Routh-Hurwitz table for the system becomes as given below.

s^3	1	$29 - 4K$
s^2	$10 + K$	$20 + 13K$
s	$-\frac{(1)(20+13K) - (10+K)(29-4K)}{10+K}$	
1	$20 + 13K$	

The Routh-Hurwitz's stability criterion implies the following conditions.

(a) $10 + K > 0$.

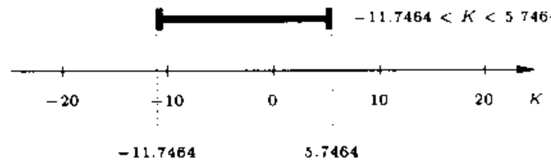
$$K > -10.$$



(b) $-\frac{(1)(20+13K) - (10+K)(29-4K)}{10+K} > 0.$

i. $10 + K > 0$ Case:

$$-\left((1)(20+13K) - (10+K)(29-4K)\right) > 0.$$
$$(20+13K) - (290 - 11K - 4K^2) < 0.$$
$$4K^2 + 24K - 270 < 0.$$
$$4(K+11.7464)(K-5.7464) < 0,$$



or

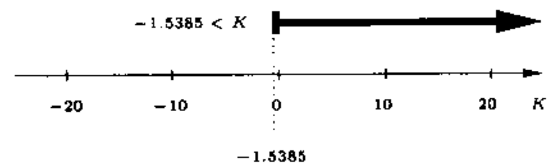
$$-11.7464 < K < 5.7464.$$

ii. $10 + K < 0$ Case:
This case results in instability from the previous condition.

(c) $20 + 13K > 0$.

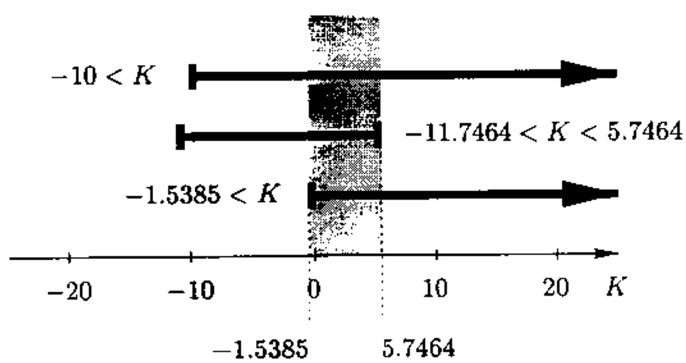
$$13K > -20.$$

$$K > -\frac{20}{13} = -1.5385.$$



The intersection of all these regions leads to

$$-1.5385 < K < 5.7464.$$



2. Consider a unity-feedback control system with the open-loop transfer function

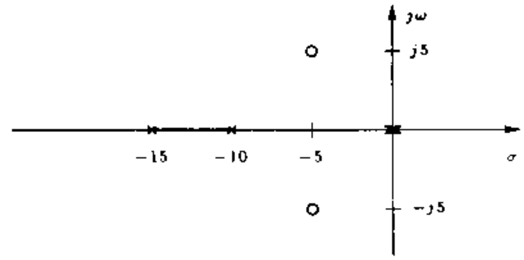
$$G(s) = K\frac{s^2+10s+50}{s^2(s+10)(s+15)} = K\frac{s^2+10s+50}{s^4+25s^3+150s^2}.$$

Construct the root-locus diagram. Determine all the important necessary features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, and angle of arrivals and departures.

Solution: First, we sketch the pole-zero locations and the real-axis portion of the root-locus diagram. Then, we decide the important features to be determined.

Need to determine:

- Asymptotes,
- Breakaway point, and
- Angle of Arrival.



There is no need to determine the imaginary axis crossings, since the branches leave the poles at $s = 0$ with the phase angle of $\pm(\pi/2)$, and the zeros pull these branches inwards.

Breakaway Point: $\dfrac{dK}{ds} = 0$

From the characteristic equation,

$$1 + G(s) = 0,$$

$$1 + K\frac{s^2 + 10s + 50}{s^4 + 25s^3 + 150s^2} = 0,$$

and

$$-K = \frac{s^4 + 25s^3 + 150s^2}{s^2 + 10s + 50}.$$

Therefore,

$$\begin{aligned} -\frac{\mathrm{d}K}{\mathrm{d}s} &= \frac{(4s^3 + 75s^2 + 300s)(s^2 + 10s + 50) - (s^4 + 25s^3 + 150s^2)(2s + 10)}{(s^2 + 10s + 50)^2}, \\ &= \frac{s(2s^4 + 55s^3 + 700s^2 + 5250s + 15000)}{(s^2 + 10s + 50)^2}. \end{aligned}$$

and for $\mathrm{d}K/\mathrm{d}s = 0$, the equation

$$s(2s^4 + 55s^3 + 700s^2 + 5250s + 15000) = 0$$

gives

$$\begin{aligned} s &= -12.4240, \\ s &= -5.4831, \\ s &= -4.7965 \pm \jmath 9.3322, \end{aligned}$$

and

$$s = 0.$$

The break-away point is the solution between -15 and -10 which is $s = -12.4240$.

Asymptotes

Real-Axis Crossing: $\sigma_a = \dfrac{\sum p_i - \sum z_i}{n-m}$

So, the real-axis crossing of the asymptotes is at

$$\begin{aligned}\sigma_a &= \frac{\sum_i p_i - \sum_i z_i}{n-m},\\ &= \frac{((0)+(0)+(-10)+(-15)) - ((-5+j5)+(-5-j5))}{4-2},\\ &= \frac{-15}{2} = -7.5.\end{aligned}$$

Real-Axis Angles: $\theta_a = \dfrac{\pm(2k+1)\pi}{n-m}$

So, the angles the asymptotes make with the real axis are determined from

$$\begin{aligned}\theta_a &= \frac{\pm(2k+1)\pi}{n-m}\\ &= \frac{\pm(2k+1)\pi}{4-2} = \pm\frac{\pi}{2}.\end{aligned}$$

Angle of Arrival: $\sum \measuredangle(\cdot) = \pm(2k+1)\pi$

The angles of arrivals to complex open-loop zeros are determined from the angular conditions about the open-loop zeros. Therefore, the angular condition about $s = -5 + j5$ is

$$\begin{aligned}&(\measuredangle(s-(-5+j5)) + \measuredangle(s-(-5-j5)))\\ &\qquad - (\measuredangle(s-(-15)) + \measuredangle(s-(-10)) + \measuredangle(s) + \measuredangle(s)) = 180^\circ + k360^\circ,\end{aligned}$$

$$\begin{aligned}&\left(\theta_{\text{arr}} + \tan^{-1}\left(\frac{(5)-(-5)}{(-5)-(-5)}\right)\right)\\ &\quad - \left(\tan^{-1}\left(\frac{(5)-(0)}{(-5)-(-15)}\right) + \tan^{-1}\left(\frac{(5)-(0)}{(-5)-(-10)}\right) + 2\tan^{-1}\left(\frac{(5)-(0)}{(-5)-(0)}\right)\right)\\ &\qquad = 180^\circ + k360^\circ,\end{aligned}$$

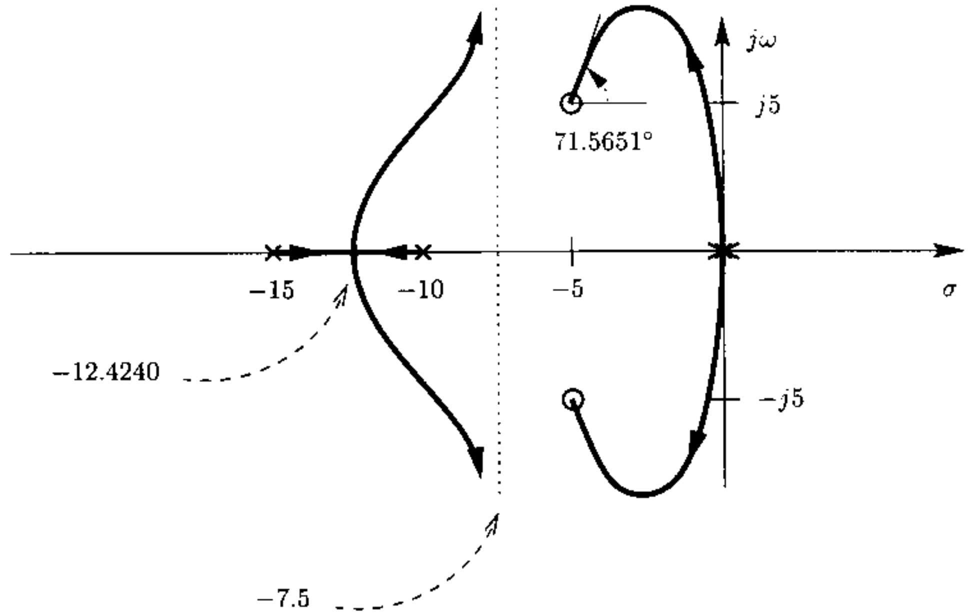
or

$$(\theta_{\text{arr}} + 90^\circ) - (26.5651^\circ + 45^\circ + 2\times 135^\circ) = 180^\circ + k360^\circ.$$

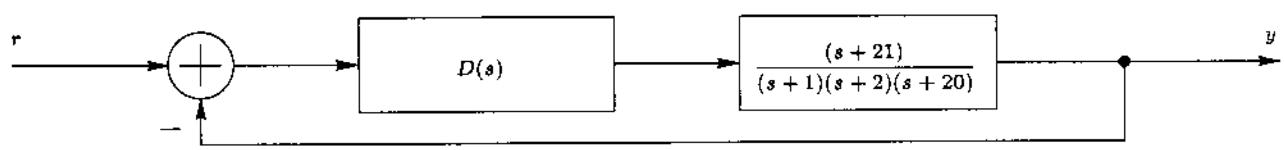
As a result,

$$\theta_{\text{arr}} = 71.5651^\circ.$$

With the features determined, we can now sketch the root-locus diagram.



3. Consider the following control system.



(a) Design a first order compensator $D(s) = D_1(s)$, such that the steady state error is zero for a step input without increasing the order of the system. If there exists a freedom of choice in the location of the compensator poles and/or zeros, choose to minimize the settling time.

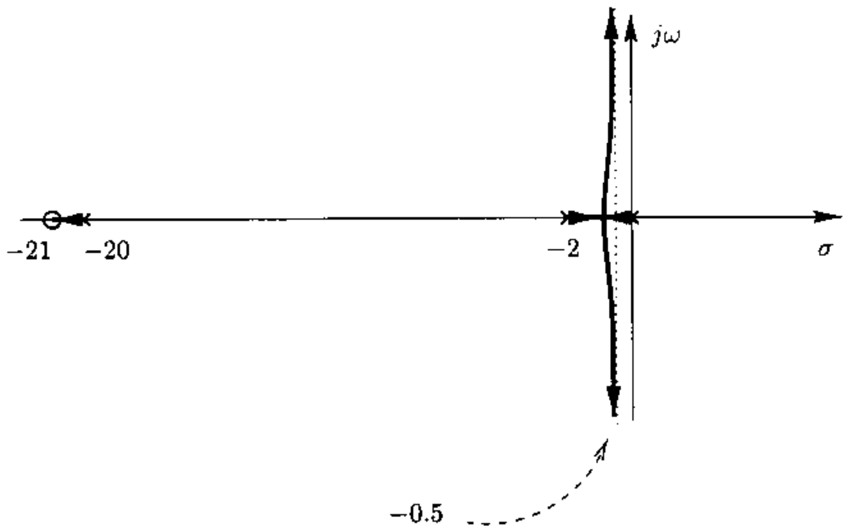
Solution: The form of the controller $D(s) = D_1(s)$ can be obtained from the desired requirements.

Given Requirements	General System Restrictions	Specific System Restrictions
The steady state error is zero for a step input without increasing the order of the system	$D_1(s) = \frac{1}{s} D_1'(s).$	For every pole or zero required in D_1, another pole or zero should be canceled.
Minimize the settling time, if there is a choice	Choose the slowest pole to cancel.	Try to cancel the poles at −1, −2, or −20 in that order.

The simplest choice from the above requirements leads to

$$D_1(s) = K\frac{s+1}{s},$$

where the resulting root-locus diagram is given below.



As we observe from the figure, the system is stable for $K > 0$. Therefore, the desired controller is

$$D(s) = D_1(s) = K\frac{s+1}{s},$$

where $K > 0$.

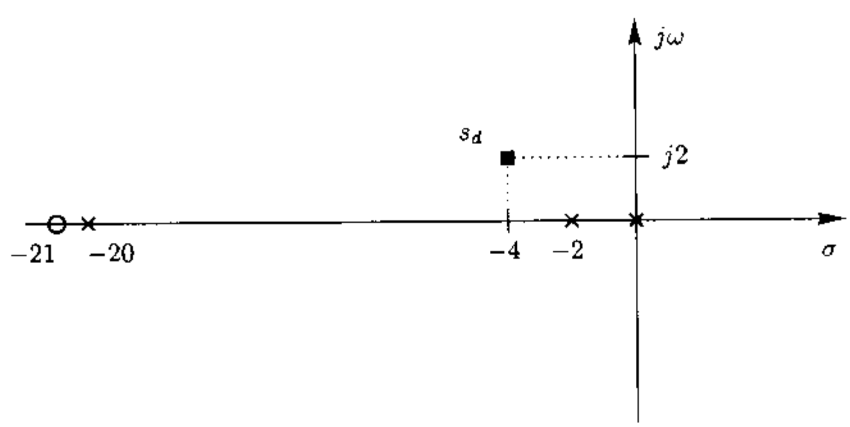
(b) Design another first order compensator $D_2(s)$ cascaded to the previous compensator, i.e. $D(s) = D_1(s)D_2(s)$, such that the desired dominant closed-loop poles are at $s_d = -4 \pm j2$, and the system order is still preserved.

Solution: For

$$D(s) = D_1(s)D_2(s) = K\frac{s+1}{s}D_2(s),$$

the open loop gain is

$$G(s)D(s) = KD_2(s)\frac{s+21}{s(s+2)(s+20)}.$$



The deficiency angle, ϕ, at s_d is calculated from the angular condition.

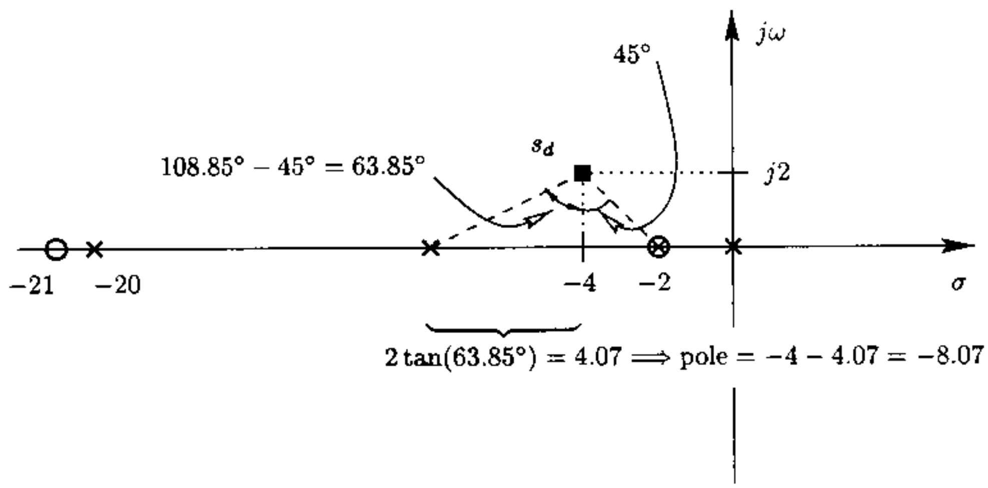
$$\phi - \measuredangle(s_d - (0)) - \measuredangle(s_d - (-2)) - \measuredangle(s_d - (-20)) + \measuredangle(s_d - (-21)) = (2k+1)\pi,$$

$$\phi - \tan^{-1}\left(\frac{(2)-(0)}{(-4)-(0)}\right) - \tan^{-1}\left(\frac{(2)-(0)}{(-4)-(-2)}\right) - \tan^{-1}\left(\frac{(2)-(0)}{(-4)-(-20)}\right) + \tan^{-1}\left(\frac{(2)-(0)}{(-4)-(-21)}\right) = 180^\circ + k360^\circ.$$

$$\phi - 153.43^\circ - 135^\circ - 7.13^\circ + 6.71^\circ = 180^\circ + k360^\circ,$$

or $\phi = 108.85^\circ$.

In order to preserve the system order, we need to cancel a pole or zero and place another one in such a way that the pole-zero combination provides the necessary deficiency angle at s_d. The best choice for cancelation is the pole at -2, since the pole at zero satisfies the steady state error requirement, and the pole at -20 and the zero at -21 are too far away.



From the above analysis,

$$D_2(s) = \frac{s+2}{s+8.07}.$$

And the magnitude K is obtained from the magnitude condition at s_d.

$$\left|G(s)D(s)\right|_{s=s_d} = 1,$$

$$\left|K\frac{s+21}{s(s+8.07)(s+20)}\right|_{s=-4+j2} = 1,$$

or $K = 19.1043$. Therefore,

$$D(s) = 19.1043\left(\frac{s+1}{s}\right)\left(\frac{s+2}{s+8.07}\right).$$

(c) Design a different compensator $D(s)$ for the original system, such that the non-zero and finite steady state error is decreased by 5 times with minimal effect on the existing closed-loop poles. Assume that the slowest physically realizable stable pole of the compensator is at -0.01.

Solution: The system is type 0 without $D(s)$, so the non-zero and finite steady-state error coefficient is K_p, and

$$e(\infty) = \frac{1}{1+K_p}.$$

Since $e_{\text{desired}}(\infty) = (1/5)e(\infty)$,

$$\frac{1}{1+K_{p_{\text{desired}}}} = \frac{1/5}{1+K_p},$$

or

$$K_{p_{\text{desired}}} = 5(1+K_p) - 1.$$

From the definition of K_p,

$$K_p = \lim_{s\to 0}\left(G(s)D(s)\right) = \frac{(21)}{(1)(2)(20)} = 0.525,$$

and $K_{p_{\text{desired}}} = 6.625$. To increase K_p to $K_{p_{\text{desired}}}$, we need to have a lag compensator with gain

$$\beta \geq \frac{K_{p_{\text{desired}}}}{K_p} = \frac{6.625}{0.525} = 12.62.$$

Assuming $\beta = 13$, the lag compensator becomes

$$D(s) = \frac{s+1/T}{s+1/(\beta T)}.$$

To have the minimal effect on the existing poles, the pole and the zero of the compensator should be as close as possible to each other. We can accomplish this necessity by choosing the pole and the zero very close to zero. Since the slowest pole of the compensator should be at -0.01; the best we can do is to choose $-1/(\beta T) = -0.01$, or $T = 7.69$. Therefore,

$$D(s) = \frac{s+0.13}{s+0.01}$$

is one possible compensator.