EE 231 **Exam#2** **Dec. 01, 2005**

75 minutes

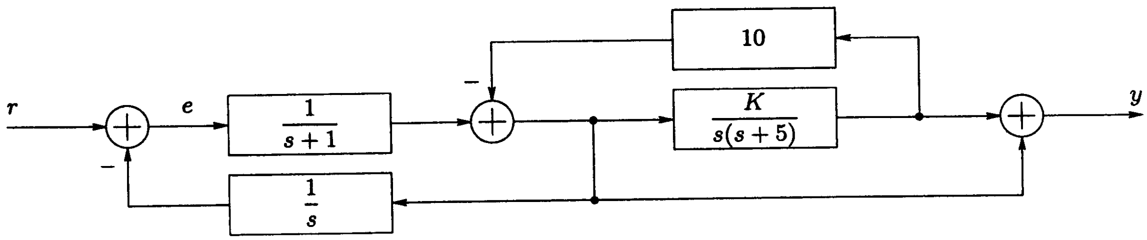
1. The following requirements are given for a second-order system that is described by the transfer function $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$.

$$\text{Maximum percent overshoot: } 10\% \leq M_p \leq 20\%.$$
$$\text{Peak time: } t_p \leq 2\,\text{s}.$$

   (a) Describe and sketch the s-plane regions of the pole locations satisfying the requirements. (15pts)

   (b) Determine the largest possible 2% settling time of a system with the poles satisfying the requirements. (10pts)

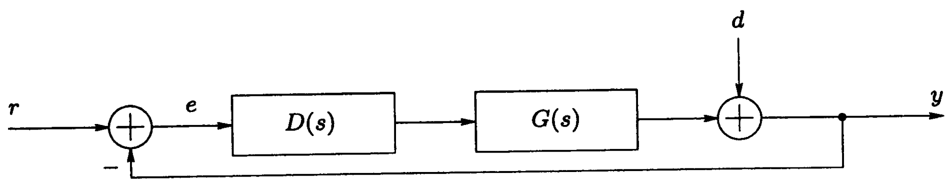
2. Consider the following control system.



   (a) Determine the set of inputs, such that the steady-state error $e(\infty)$ is zero. (10pts)

   (b) Determine the sensitivity of the transfer function with respect to K. (15pts)

3. Consider the following feedback control system with the reference input r and the disturbance input d.



   For the case when
$$G(s) = \frac{s}{s+8};$$
   design a minimal-order controller, such that the output tracks the reference input that has the laplace transform
$$R(s) = \frac{2(s-4)}{(s^2+4)(s+1)}$$
   with zero steady-state error, and a step disturbance is rejected at the output. (15pts)

4. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{s^2 - 2s + 5}{(s-1)(s+10)}.$$

(a) Construct the root-locus diagram for $K > 0$. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angle of arrivals and/or departures. (20pts)

(b) Construct the root-locus diagram for $K < 0$. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angle of arrivals and/or departures. (10pts)

(c) Determine the values of K such that the closed-loop system is stable. (05pts)

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1. The following requirements are given for a second-order system that is described by the transfer function $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$.

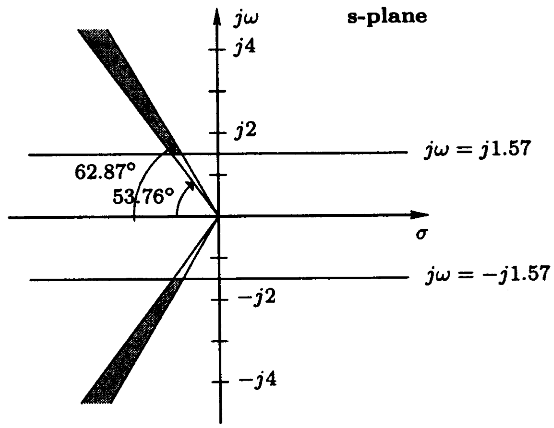
$$\text{Maximum percent overshoot: } 10\% \le M_p \le 20\%.$$
$$\text{Peak time: } t_p \le 2\,\text{s}.$$

(a) Describe and sketch the s-plane regions of the pole locations satisfying the requirements.

Solution:

Given Specifications	System Constraints	Geometrical Representations
$10\% \le M_p \le 20\%.$	$0.1 \le e^{-\left(\zeta/\sqrt{1-\zeta^2}\right)\pi} \le 0.2,$ $\frac{\lvert\ln(0.2)\rvert}{\sqrt{(\ln(0.2))^2+(\pi)^2}} \le \zeta \le \frac{\lvert\ln(0.1)\rvert}{\sqrt{(\ln(0.1))^2+(\pi)^2}},$ or $0.46 \le \zeta \le 0.59;$ since $M_p = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$, and $\zeta = \lvert\ln(M_p)\rvert/\sqrt{(\ln(M_p))^2+(\pi)^2}.$	$\cos^{-1}(0.59) \le \alpha \le \cos^{-1}(0.46)$ or $53.76^\circ \le \alpha \le 62.87^\circ$, where $\alpha = \cos^{-1}(\zeta)$ is the angle measured from the negative real axis.
$t_p \le 2\,\text{s}.$	$\frac{\pi}{\omega_d} \le 2,$ or $\omega_d \ge \pi/2;$ since $t_p = \pi/\omega_d$.	$\lvert\omega\rvert \ge \pi/2 \approx 1.57,$ since the poles are at $s = -\sigma_o \pm j\omega_d$

The shaded region describes the region specified by the given requirements.



(b) Determine the largest possible 2% settling-time of a system with the poles satisfying the requirements.

Solution: The 2% settling-time of the system is given by

$$t_{2\%s} = \frac{4}{\sigma_o}.$$

The largest settling-time is when we have the smallest σ_o. From the shaded region of the sketch in the previous part, we realize that the smallest possible real value of the poles is at the intersection of the radial line with the angle of 62.87° with respect to the negative real axis and the horizontal line at $\omega = 1.57$. From the geometry, we determine that

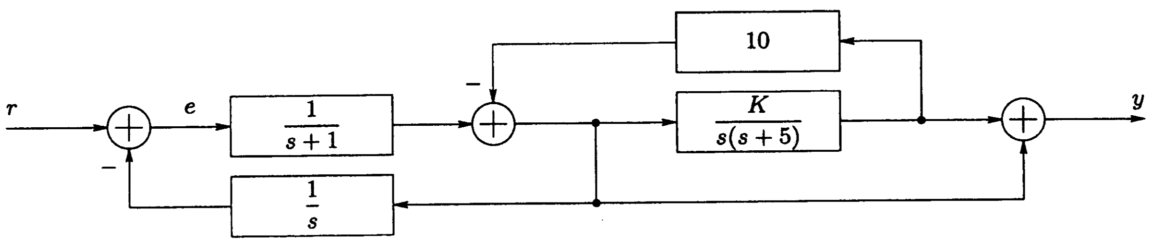
$$\tan(62.87°) = \frac{1.57}{\sigma_{o_{\min}}},$$

and

$$t_{2\%s_{\max}} = \frac{4}{\sigma_{o_{\min}}} = \frac{4}{1.57/\tan(62.87°)};$$

or the largest possible peak time of the system is 4.97 s.

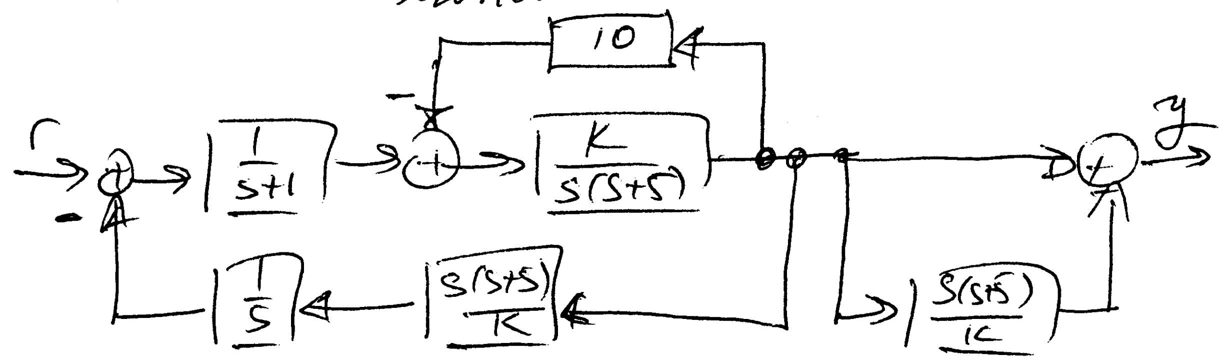
2. Consider the following control system.

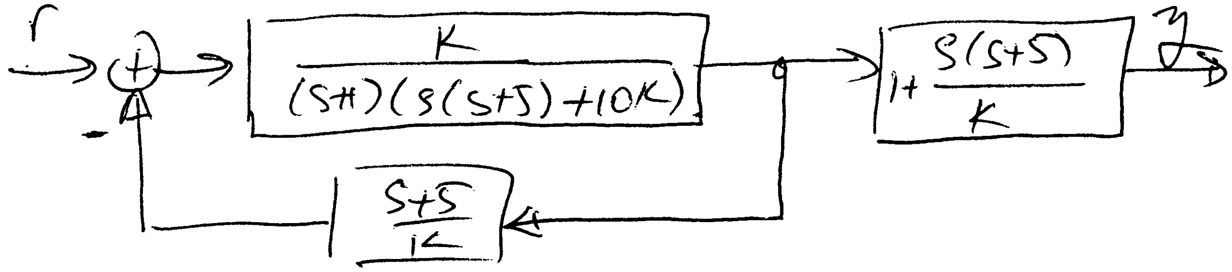


(a) Determine the set of inputs, such that the steady-state error $e(\infty)$ is zero.

(b) Determine the sensitivity of the transfer function with respect to K.

#2





(a) $G(s)H(s) = \dfrac{\cancel{K}(s+5)}{(s+1)(s^2+5s+10K)\cancel{K}}$

$e(\infty)$ is zero if (1) closed-loop system is asympt. stable
(2) (a) "r" is asympt. stable
(b) "r"'s non-asympt. stable poles are matched by $G(s)H(s)$

Closed-loop poles from

$$1 + G(s)H(s) = 0$$

$$1 + \frac{s+5}{(s+1)(s^2+5s+10K)} = 0$$

$$(s+1)(s^2+5s+10K) + (s+5) = 0$$

$$s^3 + 6s^2 + (10K+6)s + (10K+5) = 0$$

$$\begin{array}{c|cc} s^3 & 1 & 10K+6 \\ s^2 & 6 & 10K+5 \\ s & \frac{60K+36-10K-5}{6} = \frac{50K+31}{6} & \\ 1 & 10K+5 & \end{array}$$

asympt. stable, when $50K+31>0$, $\boxed{K>-\frac{1}{2}}$

$10K+5>0$, $\boxed{K>-\frac{31}{50}}$

Open-loop poles from the den. of $G(s)H(s)$

or $(s+1)(s^2+5s+10K)=0$

$s=-1$ asympt. stable

$s=-\frac{5}{2}\pm\frac{\sqrt{25-40K}}{2}$

→ non-asymp ~~unstable~~ stable when $\sqrt{25-40K}\geq 5$

$25-40K\geq 25$

$K\leq 0$

So when $-\frac{1}{2}<K\leq 0$, the open-loop system will have a non-asymp. stable pole at $s=(\sqrt{25-40K}-5)/2$

$\boxed{\text{So "R(s)" can have asymptotically stable poles and up to one non-asympt. stable pole at } s=K \text{ when } -\frac{1}{2}<K\leq 0 \text{ as long as } K>-\frac{1}{2}.}$

5)
$$\frac{Y(s)}{R(s)} = \frac{K/(s+1)(s^2+5s+10K)}{1+(s+5)/(s+1)(s^2+5s+10K)}\left(1+\frac{s(s+5)}{K}\right)$$

$$= \frac{1}{(s+1)(s^2+5s+10K)+(s+5)}\left(K+s(s+5)\right)$$

$$= \frac{s^2+5s+K}{s^3+6s^2+(10K+6)s+(10K+5)}$$

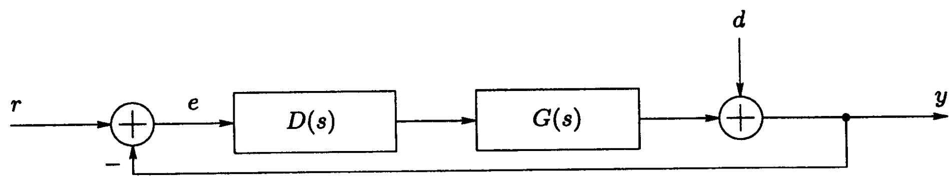
$$\frac{d\,Y(s)/R(s)}{dK} = \frac{(s^3+6s^2+(10K+6)s+(10K+5))-(s^2+5s+K)(10s+10)}{(s^3+6s^2+(10K+6)s+(10K+5))^2}$$

$$= \frac{-9s^3-54s^2-44s+5}{(\ -\ -\ -\ -\ -\ -\ )^2}$$

$$S_K^{Y/R} = \frac{K}{Y/R}\,\frac{dY/R}{dK}$$

$$\boxed{S_K^{Y/R} = \frac{K(-9s^3+54s^2-44s+5)}{(s^2+5s+K)(s^3+6s^2+(10K+6)s+(10K+5))}}$$

3. Consider the following feedback control system with the reference input r and the disturbance input d.



For the case when

$$G(s) = \frac{s}{s+8};$$

design a minimal-order controller, such that the output tracks the reference input that has the laplace transform

$$R(s) = \frac{2(s-4)}{(s^2+4)(s+1)}$$

with zero steady-state error, and a step disturbance is rejected at the output.

Solution: In order to have a zero steady-state error for any given input and to reject a step disturbance at the output, we need to match the non-asymptotically stable poles of the input and the disturbance in the open-loop gain of the system. In the case of the given input, we need to have poles at $s = \pm j2$; since the pole at $s = -1$ of $R(s)$ is asymptotically stable, and its contribution will disappear on its own at steady state. To reject a step disturbance, we also need to match the disturbance pole at $s = 0$, or the system has to be of type-1. However, in this case the gain $G(s)$ has a zero at $s = 0$, and a simple pole at $s = 0$ would be canceled by this zero to end up with a type-0 system. Therefore, we need to have two poles at $s = 0$ supplied by the controller, such that the open-loop gain

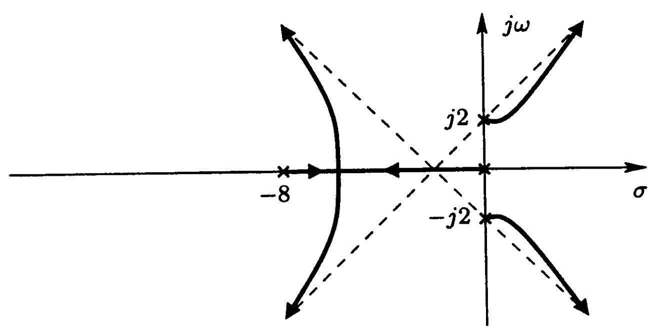
$$D(s)G(s) = \left(\frac{1}{s^2(s^2+4)}\,D'(s)\right)\left(\frac{s}{s+8}\right) = \frac{1}{s(s^2+4)(s+8)}D'(s),$$

has a pole left at $s = 0$, where

$$D(s) = \frac{1}{s^2(s^2+4)}D'(s)$$

for some $D'(s)$. Since there is no other explicit requirement, we only need to ensure stability by a proper and simple choice of $D'(s)$.

The simplest choice is $D'(s) = K$ for a constant K. We may use a number of methods to check the stability of the system for this choice, but a rough sketch of the root-locus, as shown below, is simple enough to see the location of the closed-loop poles.



As we observe from the root-locus diagram, there is no value of K that would result in a stable closed-loop system; mainly because the asymptote angles are $\theta_a = \pm 45°$, $\pm 135°$, and there are poles on the imaginary axis.

In order to have the asymptote intersection and the angles stay inside the left-half plane, we need to have zeros in $D'(s)$. Since we are placing four poles, we may have up to four zeros in $D'(s)$. With only one zero, we will still have a similar problem, since the asymptote angles will be $\theta_a = \pm 60°$, $180°$. As a result, we need to place two zeros to generate a case, where the asymptote angles are $\theta_a = \pm 90°$, and the asymptote intersection is on the left-half plane. For

$$D'(s) = K(s+a)(s+b),$$

or

$$D(s)G(s) = K\frac{(s+a)(s+b)}{s(s^2+4)(s+8)},$$

the asymptote intersection

$$\sigma_a = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{((-8)+(0)+(j2)+(-j2)) - ((-a)+(-b))}{4-2} = \frac{a+b-8}{2},$$

where $\sum p_i$ and $\sum z_i$ are the sums of the pole and zero locations, respectively. As long as $a+b<8$, we get $\sigma_a < 0$.

Therefore, one possible simplest controller is

$$D(s) = K\frac{(s+a)(s+b)}{s^2(s^2+4)},$$

where $a+b<8$, and $K>0$.

4. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{s^2-2s+5}{(s-1)(s+10)}.$$

(a) Construct the root-locus diagram for $K>0$. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angle of arrivals and/or departures.

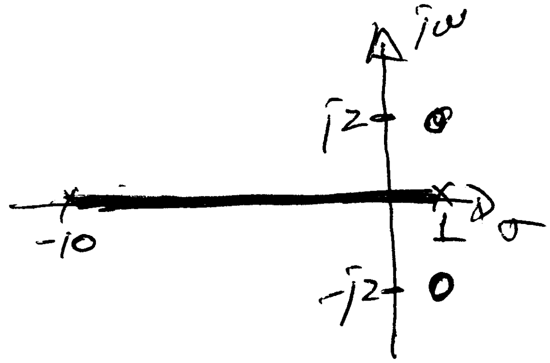
(b) Construct the root-locus diagram for $K<0$. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angle of arrivals and/or departures.

(c) Determine the values of K such that the closed-loop system is stable.

#4

$$G(s) = K\,\frac{s^2-2s+5}{(s-1)(s+10)} = \frac{(s-(1+j2))(s-(1-j2))}{(s-1)(s+10)}$$

ⓐ $K>0$



Real-axis

Need to determine
- Break-away
- Imaginary-axis crossings
- Angle of arrival

Break-away

$$1+G(s)=0 \Rightarrow 1+K\,\frac{s^2-2s+5}{(s-1)(s+10)}=0$$

$$-K = \frac{(s-1)(s+10)}{s^2-2s+5} = \frac{s^2+9s+10}{s^2-2s+5}$$

$$-\frac{dK}{ds} = \frac{(2s+9)(s^2-2s+5)-(s^2+9s+10)(2s-2)}{(s^2-2s+5)^2}$$

$$\frac{dK}{ds}=0 \Rightarrow \frac{-11s^2+30s+25}{(s-2s+5)^2}=0 \text{ or } \begin{array}{l} s=-0.669153 \\ s=3.39643 \end{array}$$

for $K>0$ (for $s=-0.669153$)

for $K<0$ (for $s=3.39643$)

Imaginary-axis crossings

$$1+G(s)=0 \Rightarrow (s-1)(s+10)+K(s^2-2s+5)=0$$

$$(K+1)s^2+(-2K+9)s+(5K-10)=0$$

$$\begin{array}{c|cc} s^2 & K+1 & 5K-10 \\ s & -2K+9 & \\ 1 & 5K-10 & \end{array}$$

From s-term $-2K+9=0 \Rightarrow K=9/2$

Put to s^2-term $\left[(K+1)s^2+(5K-10)\right]_{K=9/2}=0$

$$\frac{11}{2}s^2+\frac{45-20}{2}=0$$

$$s=\pm j\,5/\sqrt{11}=\pm j\frac{5\sqrt{11}}{11}\approx j1.5$$

Angle of arrival

$\measuredangle$ condition about $s=1+j2$

$$-\measuredangle(s-(-10))-\measuredangle(s-(1))$$

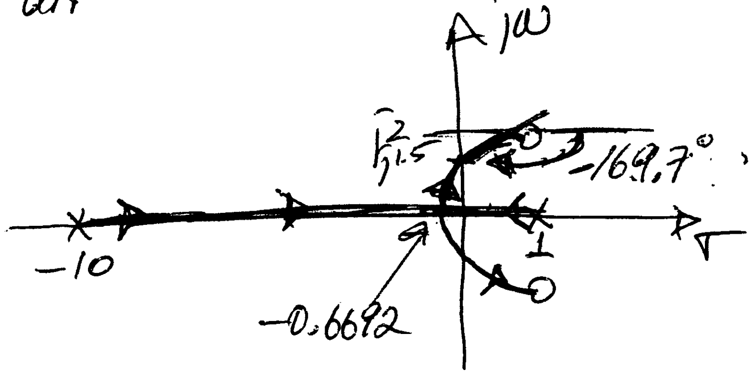
$$+\measuredangle(s-(1-j2))+\phi_{arr}=180^\circ+k360^\circ$$

$$-\tan^{-1}\left(\frac{(2)-(0)}{(1)-(-10)}\right)-\tan^{-1}\left(\frac{(2)-(0)}{(1)-(1)}\right)$$

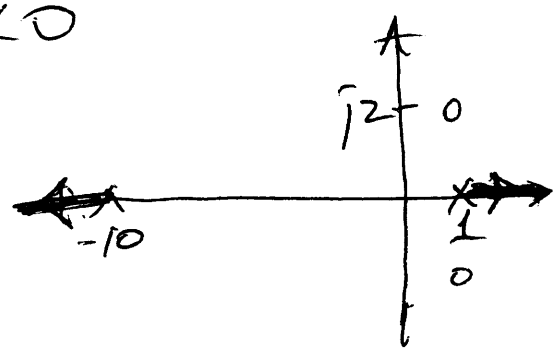
$$+\tan^{-1}\left(\frac{(2)-(-1)}{(1)-(1)}\right)+\phi_{arr}=180^\circ+k360^\circ$$

$$-10.3^\circ-90^\circ+90^\circ+\phi_{arr}=180^\circ+k360^\circ$$

$$\phi_{arr}=-169.7^\circ$$



(b) $K<0$



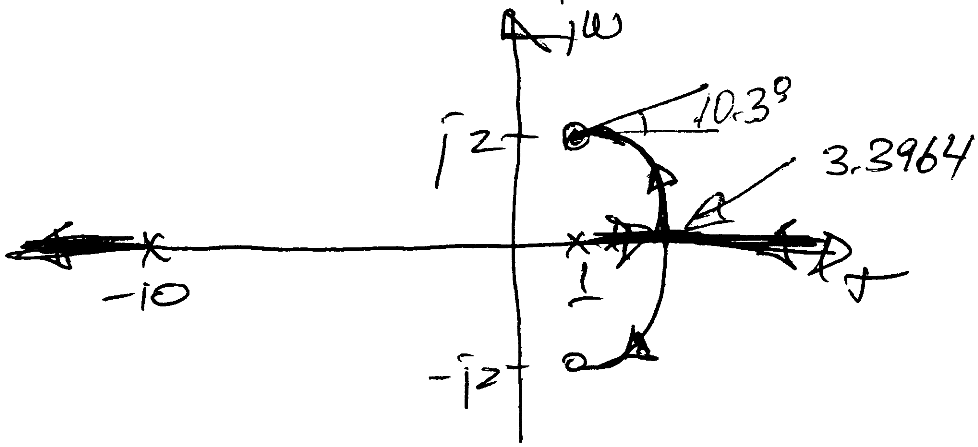
From prev. part break-away pt ū at $s=3.39643$
& no imaginary axis crossing.

<u>Angle of arrival</u> ū similar

$$-\angle(s-(-10))-\angle(s-1)$$
$$+\angle(s-(1-j2))+\phi_{arr}=k360^\circ$$

$$-10.3^\circ - 90^\circ + 90^\circ + \phi_{arr} = k360^\circ$$

$$\phi_{arr} = 10.3^\circ$$



(c) From the Routh-Hurwitz table

$$\left.\begin{array}{lll} K+1>0 & \Rightarrow & K>1 \\ -2K+9>0 & \Rightarrow & K<9/2 \\ 5K-10>0 & \Rightarrow & K>2 \end{array}\right\} \boxed{2<K<9/2}$$