EE 231 Exam#2 75 minutes Dec. 05, 2006

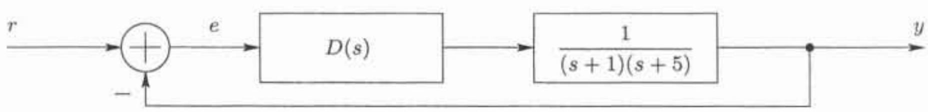
1. The following requirements are given for a second-order system that is described by the transfer function $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$.

$$\begin{aligned} \text{Maximum percent overshoot:} &\quad 5\% \le M_p \le 15\%. \\ \text{Peak time:} &\quad t_p \le 1\,\text{s}. \\ 2\% \text{ settling time:} &\quad t_{2\%s} \le 2\,\text{s}. \end{aligned}$$

   (a) Describe and sketch the s-plane regions of the pole locations satisfying the requirements. (15pts)

   (b) Determine the largest possible rise time of a system with the poles satisfying the requirements. (10pts)

2. Consider the following feedback control system.

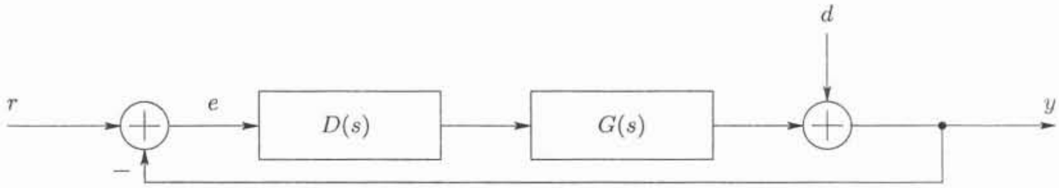


Design a proportional-integral (PI) controller

$$D(s) = K_P + \frac{K_I}{s} = \frac{K_P s + K_I}{s} = K_P \frac{s + (K_I/K_P)}{s};$$

such that the 2% settling-time is less than 2 seconds, and the steady-state error for the unit-ramp input $e(\infty) = (1/2)$. (25pts)

3. Consider the following feedback control system with the reference input r and the disturbance input d.



For the case when

$$G(s) = \frac{s+5}{s+8},$$

design a minimal-order controller, such that the output tracks the reference input that has the laplace transform

$$R(s) = \frac{2(s-4)}{(s^2+4)(s+1)}$$

with zero steady-state error, and a step disturbance is rejected at the output. (25pts)

4. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{(s+2)^2}{s^3} = K\frac{s^2+4s+4}{s^3}.$$

Construct the root-locus diagram. Determine all the important necessary features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, and angle of arrivals and departures. (25pts)

EE 231 — Exam#2 Solutions — Dec. 05, 2006

1. The following requirements are given for a second-order system that is described by the transfer function $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$.

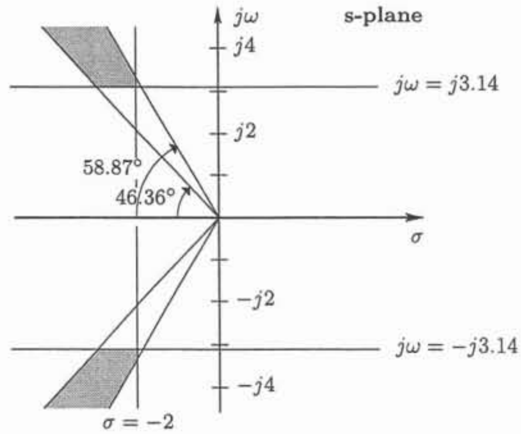
$$\begin{aligned} \text{Maximum percent overshoot:} &\quad 5\% \le M_p \le 15\%. \\ \text{Peak time:} &\quad t_p \le 1\,\text{s}. \\ 2\% \text{ settling time:} &\quad t_{2\%s} \le 2\,\text{s}. \end{aligned}$$

(a) Describe and sketch the s-plane regions of the pole locations satisfying the requirements.

Solution:

Given Specifications	System Constraints	Geometrical Representations
$5\% \le M_p \le 15\%.$	$0.05 \le e^{-\left(\zeta/\sqrt{1-\zeta^2}\right)\pi} \le 0.15,$ $\frac{\lvert\ln(0.15)\rvert}{\sqrt{(\ln(0.15))^2 + (\pi)^2}} \le \zeta \le \frac{\lvert\ln(0.05)\rvert}{\sqrt{(\ln(0.05))^2 + (\pi)^2}},$ or $0.51 \le \zeta \le 0.69;$ since $M_p = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$, and $\zeta = \lvert\ln(M_p)\rvert/\sqrt{(\ln(M_p))^2 + (\pi)^2}.$	$\cos^{-1}(0.69) \le \alpha \le \cos^{-1}(0.51)$ or $46.36^\circ \le \alpha \le 58.87^\circ,$ where $\alpha = \cos^{-1}(\zeta)$ is the angle measured from the negative real axis.
$t_p \le 1\,\text{s}.$	$\frac{\pi}{\omega_d} \le 1,$ or $\omega_d \ge \pi/1;$ since $t_p = \pi/\omega_d.$	$\lvert\omega\rvert \ge \pi \approx 3.14,$ since the poles are at $s = -\sigma_o \pm j\omega_d$
$t_{2\%s} \le 2\,\text{s}.$	$\frac{4}{\sigma_o} \le 2,$ or $\sigma_o \ge 2;$ since $t_{2\%s} = 4/\sigma_o.$	$\sigma \le -2,$ since the poles are at $s = -\sigma_o \pm j\omega_d$

The shaded region describes the region specified by the given requirements.



(b) Determine the largest possible rise time of a system with the poles satisfying the requirements.

Solution: The rise time of the system is given by

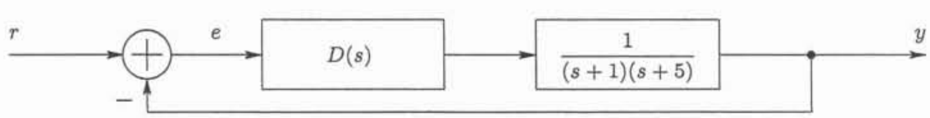
$$t_r = \frac{\pi - \cos^{-1}(\zeta)}{\omega_d}.$$

The largest rise time is when we have the largest $\left(\pi - \cos^{-1}(\zeta)\right)$ or the smallest $\cos^{-1}(\zeta)$ and the smallest ω_d. From the shaded region of the sketch in the previous part, we realize that the smallest ω_d is when $\omega_d = \pi \approx 3.14$ and the smallest $\cos^{-1}(\zeta)$ is when $\cos^{-1}(\zeta) = \cos^{-1}(0.69) = 46.36° = 0.2576\pi$, which is at the intersection of the radial line with the angle of 46.36° with respect to the negative real axis and the horizontal line at $\omega = 3.14$. Therefore,

$$t_{r_{\max}} = \frac{\pi - \cos^{-1}(\zeta_{\max})}{\omega_{d_{\min}}} = \frac{\pi - 0.2576\pi}{\pi};$$

or the largest possible rise time of the system is 0.74 s.

2. Consider the following feedback control system.



Design a proportional-integral (PI) controller

$$D(s) = K_P + \frac{K_I}{s} = \frac{K_P s + K_I}{s} = K_P \frac{s + (K_I/K_P)}{s};$$

such that the 2% settling-time is less than 2 seconds, and the steady-state error for the unit-ramp input $e(\infty) = (1/2)$.

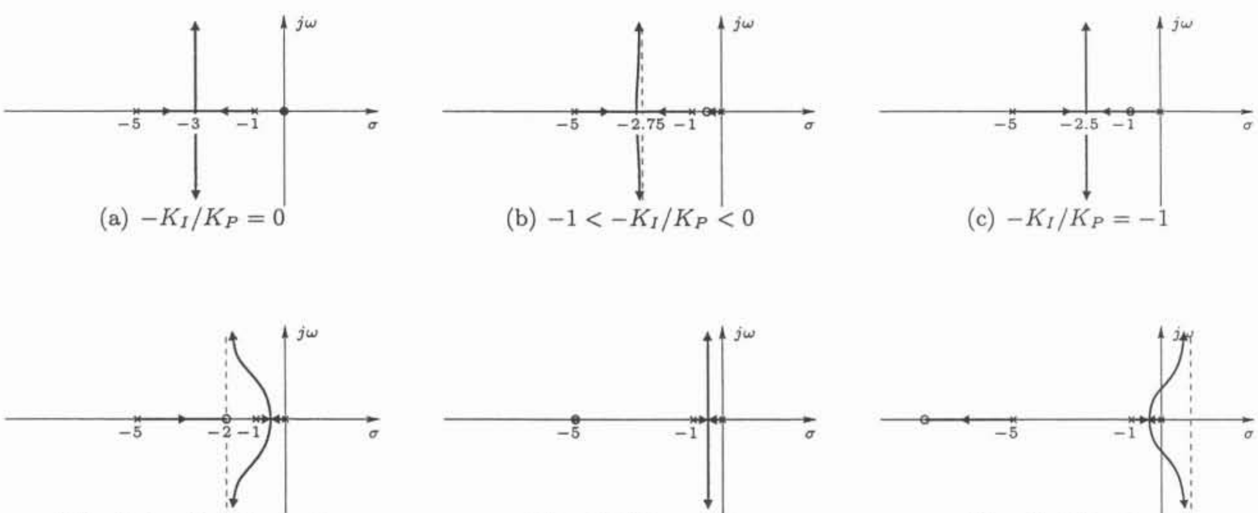
Solution: Since the open-loop gain of the system is

$$D(s)G(s) = K_P \left(\frac{s + (K_I/K_P)}{s}\right) \left(\frac{1}{(s+1)(s+5)}\right),$$

the system is type-1, and we can get a constant steady-state error for a ramp input. The 2% settling-time requirement gives us the restriction such that

$$t_{2\%s} = \frac{4}{\sigma_o} \le 2,$$

or $\sigma_o \ge 2$. In other words, the real part of the dominant complex closed-loop poles or the dominant real pole needs to be less than -2. In the PI design, we have the choice of a zero and the loop gain. Since we have three open-loop gain poles at $s = 0$, -1, and -5, as well as a zero at $s = -K_I/K_P$; we get the following possible root-locus diagrams.



(a) $-K_I/K_P = 0$ (b) $-1 < -K_I/K_P < 0$ (c) $-K_I/K_P = -1$

(d) $-5 < -K_I/K_P < -1$ (e) $-K_I/K_P = -5$ (f) $-K_I/K_P < -5$

In case (a), the system is no longer type-1. In case (b), the dominant closed-loop real pole is greater than -1. In the cases (d)–(f), the dominant poles all have real parts greater than -2. So the only good option is to cancel the pole at -1 as in the case (c). In other words,

$$D(s) = K_P \frac{s + (K_I/K_P)}{s} = K_P \frac{s+1}{s}$$

satisfies the settling-time requirement for the values of K_P after the root-locus branch crosses $s = -2$. When $s = -2$, from the magnitude condition we have

$$\left|D(s)G(s)\right|_{s=-2} = \left|K_P \left(\frac{s+1}{s}\right)\left(\frac{1}{(s+1)(s+5)}\right)\right|_{s=-2} = \left|K_P \frac{1}{s(s+5)}\right|_{s=-2} = 1,$$

or $K_P = 6$. So, as long as we choose $K_P \ge 6$, we satisfy the settling-time requirement.

The steady-state error for the unit-ramp input is $e(\infty) = (1/K_v)$, where

$$K_v = \lim_{s\to 0} sD(s)G(s) = \lim_{s\to 0} sK_P \frac{1}{s(s+5)} = \frac{K_P}{5}.$$

Since we need $e(\infty) = (1/2)$,

$$e(\infty) = \frac{1}{K_v} = \frac{1}{K_P/5} = \frac{5}{K_P} = \frac{1}{2},$$

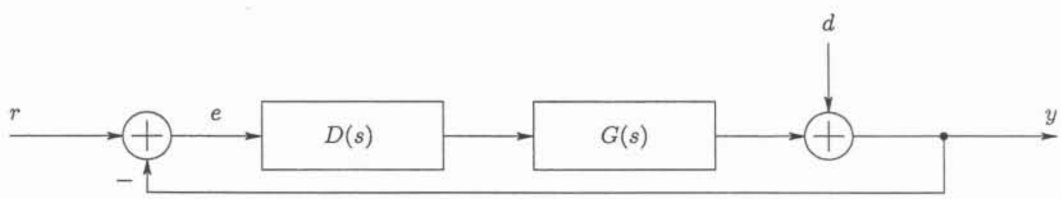
or $K_P = 10$ will satisfy the steady-state error requirement as well as the settling-time requirement. Therefore,

$$D(s) = 10\,\frac{s+1}{s},$$

or

$$D(s) = 10 + \frac{10}{s}.$$

3. Consider the following feedback control system with the reference input r and the disturbance input d.



For the case when

$$G(s) = \frac{s+5}{s+8},$$

design a minimal-order controller, such that the output tracks the reference input that has the laplace transform

$$R(s) = \frac{2(s-4)}{(s^2+4)(s+1)}$$

with zero steady-state error, and a step disturbance is rejected at the output.

Solution: In order to have a zero steady-state error for any reference input and to reject a disturbance signal at the output, we need to match the non-asymptotically stable poles of the input and the disturbance in the open-loop gain of the system. In the case of the reference input, we need to have poles at $s = \pm j2$; since the pole at $s = -1$ of $R(s)$ is asymptotically stable, and its contribution will disappear on its own at steady state. To reject a step disturbance, we also need to match the disturbance pole at $s = 0$, or the system has to be of type-1. With these choices, the open-loop gain

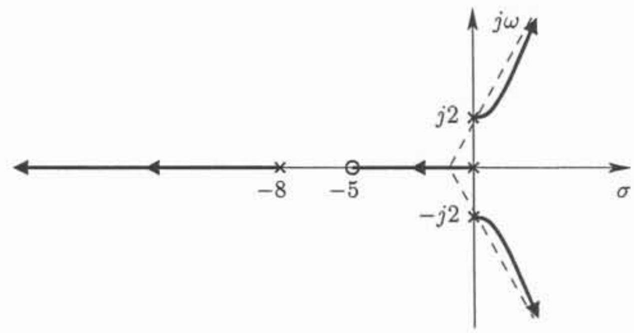
$$D(s)G(s) = \left(\frac{1}{s(s^2+4)}\,D'(s)\right)\left(\frac{s+5}{s+8}\right) = \frac{s+5}{s(s^2+4)(s+8)}D'(s),$$

where

$$D(s) = \frac{1}{s(s^2+4)}D'(s)$$

for some $D'(s)$. Since there is no other explicit requirement, we only need to ensure stability by a proper and simple choice of $D'(s)$.

The simplest choice is $D'(s) = K$ for a constant K. We may use a number of methods to check the stability of the system for this choice, but a rough sketch of the root-locus, as shown below, is simple enough to see the location of the closed-loop poles.



As we observe from the root-locus diagram, there is no value of K that would result in a stable closed-loop system; mainly because the asymptote angles are $\theta_a = \pm 60°$, $180°$, and there are poles on the imaginary axis.

In order to have the asymptote intersection and the angles stay inside the left-half plane, we need to have zeros in $D'(s)$. Since we are placing three poles, we may have up to three zeros in $D'(s)$. With only one zero, we will be able to have the asymptote angle as $\theta_a = \pm 90°$. We need to make sure that the asymptote intersection is on the left-half plane. For

$$D'(s) = K(s-a),$$

or

$$D(s)G(s) = K\frac{(s-a)(s+5)}{s(s^2+4)(s+8)},$$

the asymptote intersection

$$\sigma_a = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{\big((-8)+(0)+(j2)+(-j2)\big) - \big((a)+(-5)\big)}{4-1} = \frac{-a-3}{3},$$

where $\sum p_i$ and $\sum z_i$ are the sums of the pole and zero locations, respectively. As long as $a > -3$, we get $\sigma_a < 0$, and the complex poles will go towards the asymptotically stable region. However, if $a > 0$; this time the pole at $s = 0$ will go towards the zero at $s = -a$, and the system will still be unstable. So, we need to choose $-3 < a < 0$ and $K \gg 0$. (For small $K > 0$, there might be a region of the root-locus branch that is still in the unstable region.)

Therefore, one possible simplest controller is

$$D(s) = K\frac{(s-a)}{s(s^2+4)},$$

where $-3 < a < 0$ and $K \gg 0$.

4. Consider a negative unity-feedback control system with the open-loop transfer function

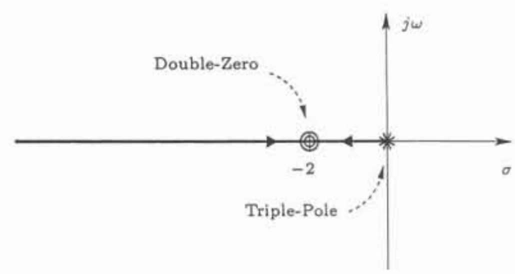
$$G(s) = K\frac{(s+2)^2}{s^3} = K\frac{s^2+4s+4}{s^3}.$$

Construct the root-locus diagram. Determine all the important necessary features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, and angle of arrivals and departures.

Solution: First, we sketch the pole-zero locations and the real-axis portion of the root-locus diagram. Then, we decide the important features to be determined.

Need to determine:

- Break-in points, and
- Imaginary-axis crossings.



There is only one asymptote, since two out of the three poles will go towards the double zero. The initial break-away from the triple poles at $s = 0$ will be at $\pm 60°$ and $180°$, since they will leave the real axis equally apart and one has to leave at $180°$.

Break-in Point: $\mathrm{d}K/\mathrm{d}s = 0$

From the characteristic equation,

$$1 + G(s) = 0,$$

$$1 + K\frac{(s+2)^2}{s^3} = 0,$$

and

$$-K = \frac{s^3}{(s+2)^2}.$$

Therefore,

$$-\frac{\mathrm{d}K}{\mathrm{d}s} = \frac{3s^2(s+2)^2 - s^3(2(s+2))}{(s+2)^4};$$

and for $\mathrm{d}K/\mathrm{d}s = 0$, the equation

$$3s^2(s+2)^2 - s^3(2(s+2)) = s^2(s+2)(3(s+2) - 2s) = s^2(s+2)(s+6) = 0$$

gives $s = -6$, $s = -2$, $s = 0$, and $s = 0$. Since the double $s = 0$ location is the initial break-away, and the $s = -2$ location is due to the double zero, the break-in point is at $s = -6$.

Imaginary-Axis Crossings: Routh-Hurwitz Table

The imaginary axis crossings can be determined from the Routh-Hurwitz table. The characteristic equation from

$$1 + G(s) = 1 + K\frac{s^2 + 4s + 4}{s^3} = \frac{s^3 + K(s^2 + 4s + 4)}{s^3} = 0$$

is

$$s^3 + Ks^2 + 4Ks + 4K = 0.$$

The Routh-Hurwitz table for this characteristic equation is given below.

s^3	1	$4K$
s^2	K	$4K$
s	$\dfrac{4K^2 - 4K}{K} = 4K - 4$	
1	$4K$	

The imaginary-axis crossings will correspond to the values of K that would make a row of all zeros on the table. When $K = 0$, the s^2-row and the 1-row become all zeros, because there are multiple imaginary-axis crossings at the start of the root-locus diagram. The only other candidate is the s-row. The s-row is all zero, when $K = 1$. For this value of K, we get a factor of the characteristic polynomial from the upper or the s^2-row. So,

$$\left(Ks^2 + 4K\right)_{K=1} = s^2 + 4 = 0,$$

or $s = \pm j2$. Therefore, the imaginary-axis crossings are at $s = \pm j2$.

With the features determined, we can now sketch the root-locus diagram.