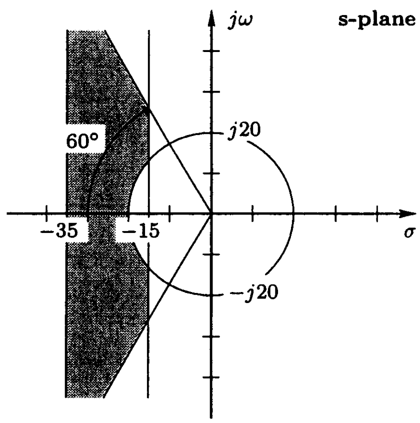
EE 231 | Exam#2 | Apr. 28, 2005

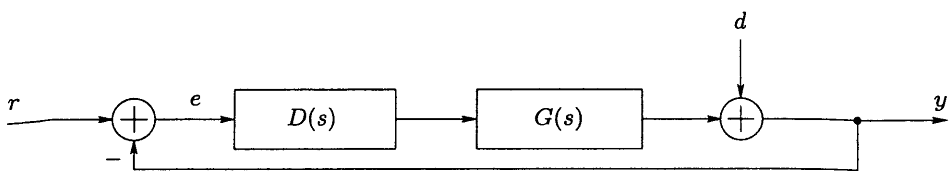
75 minutes

1. (a) Obtain the necessary inequalities to describe the strictly complex poles in the shaded region below in terms of only ζ and ω_n of a second-order system described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$. (15pts)



(b) Consider a second-order system with no zero, such that its poles are located in the shaded region above. Determine the largest possible maximum percent-overshoot and the largest possible 2% settling-time of the system. (10pts)

2. Consider the following feedback control system with the reference input r and the disturbance input d.



For the case, when

$$G(s) = \frac{3(s+4)}{s(s+8)};$$

design a minimal-order controller, such that the output tracks the reference input that has the laplace transform

$$R(s) = \frac{2(s+6)}{(s^2+4)(s+2)}$$

with zero steady-state error, and a step disturbance is rejected at the output. (15pts)

3. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{(s-2)(s+1)}{(s^2-4s+29)(s^2+6s+10)} = K\frac{s^2-s-2}{s^4+2s^3+15s^2+134s+290}.$$

(a) Determine the values of K such that the closed-loop system is asymptotically stable. (20pts)

(b) Determine the value (or values) of K and the natural frequency (or frequencies), such that the closed-loop system would have sustained oscillations. (10pts)

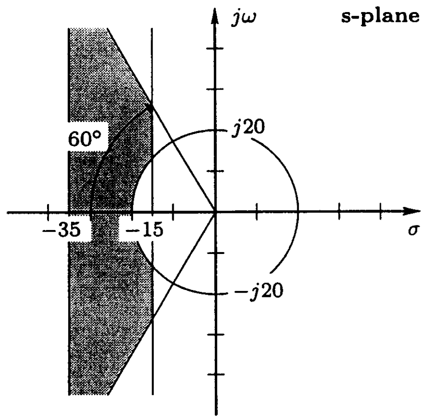
4. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{s^2-4s+8}{s(s^2+2s+2)(s+10)} = K\frac{s^2-4s+8}{s^4+12s^3+22s^2+20s}.$$

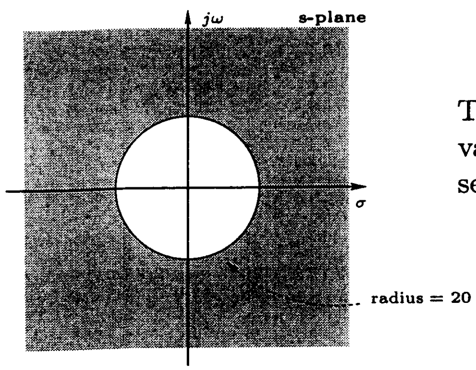
Construct the root-locus diagram. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angle of arrivals and/or departures. (30pts)

EE 231 Exam#2 Solutions Apr. 28, 2005

1. (a) Obtain the necessary inequalities to describe the strictly complex poles in the shaded region below in terms of only ζ and ω_n of a second-order system described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$.

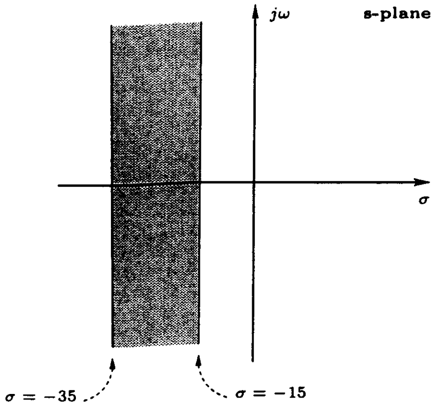


Solution: To be able to describe the shaded region, we need to separate it into unions or intersections of simpler regions.



The equi-distance points from the origin designate constant value for ω_n. As a result, the shown shaded area is represented by

$$\omega_n \geq 20.$$

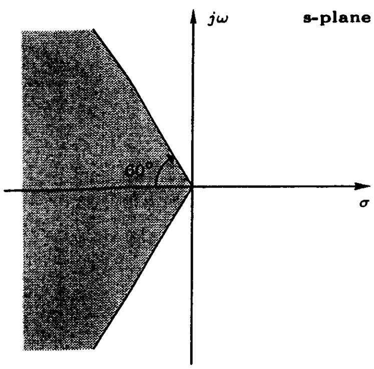


A vertical straight line designates a constant value for the real part of the poles. Since the real part of the complex poles are at $-\zeta\omega_n$, the shown shaded area is represented by

$$-35 \leq -\zeta\omega_n \leq -15,$$

or

$$15 \leq \zeta\omega_n \leq 35.$$



A straight line originating from the origin designates a constant ζ value, where $\cos^{-1}(\zeta)$ is the acute angle between the line and the negative real axis. So for the shaded area shown, we have

$$\cos^{-1}(\zeta) \leq 60°,$$

or

$$\zeta \geq \cos(60°),$$

since $\cos(\theta)$ is a monotonically decreasing function for $0 < \theta < 180°$. So, we have

$$\zeta \geq \frac{1}{2}.$$

Therefore, the shaded area given in the problem is the intersection of the individual shaded areas, and it can be represented by

$$\omega_n \geq 20,$$

$$15 \leq \zeta\omega_n \leq 35,$$

$$\zeta \geq 1/2.$$

(b) Consider a second-order system with no zero, such that its poles are located in the shaded region above. Determine the largest possible maximum percent-overshoot and the largest possible 2% settling-time of the system.

Solution: Maximum overshoot for a second-order system with no zero is given by

$$M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}.$$

The only system parameter that affects the maximum overshoot is ζ. For maximum M_p, we need to have minimum ζ; since $\zeta = 0$ gives undamped oscillations. In the shaded region, the minimum $\zeta = 1/2$, and the corresponding maximum overshoot is

$$M_p = e^{-\frac{1/2}{\sqrt{1-(1/2)^2}}\pi} = e^{-\pi/\sqrt{3}} \approx 0.1630,$$

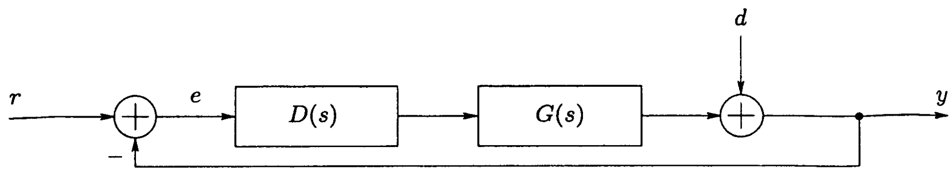
or the largest possible maximum percent-overshoot is 16.3%.

The 2% settling time of a second-order system with no zero is given by

$$t_{2\%s} = \frac{4}{\sigma_o} = \frac{4}{\zeta\omega_n}.$$

The only system parameter that affects the settling time is σ_o. For maximum $t_{2\%s}$, we need to have minimum σ_o. In the shaded region, the minimum $\sigma_o = 15$, and as a result the largest possible 2% settling time is 4/15 s.

2. Consider the following feedback control system with the reference input r and the disturbance input d.



For the case, when

$$G(s) = \frac{3(s+4)}{s(s+8)};$$

design a minimal-order controller, such that the output tracks the reference input that has the laplace transform

$$R(s) = \frac{2(s+6)}{(s^2+4)(s+2)}$$

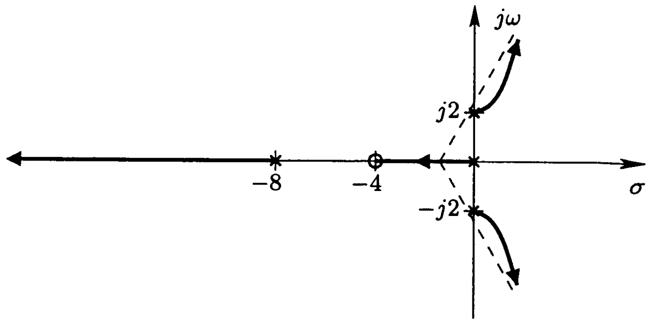
with zero steady-state error, and a step disturbance is rejected at the output.

Solution: In order to have a zero steady-state error for any given input and to reject a step disturbance at the output, we need to match the non-asymptotically stable poles of the input and the disturbance in the open-loop gain of the system. In the case of the given input, we need to have poles at $s = \pm j2$; since the pole at $s = -2$ of $R(s)$ is asymptotically stable, and its contribution will disappear on its own in steady state. To reject a step disturbance, we also need to match the disturbance pole at $s = 0$, or the system has to be of type-1. However, in this case the gain $G(s)$ already has a pole at $s = 0$. Therefore, we only need to have poles at $s = \pm j2$ supplied by the controller, such that the open-loop gain is

$$D(s)G(s) = \left(\frac{1}{s^2+4}D'(s)\right)\left(\frac{3(s+4)}{s(s+8)}\right) = \frac{3(s+4)}{s(s^2+4)(s+8)}D'(s)$$

for some $D'(s)$. Since there is no other explicit requirement, we only need to ensure stability by a proper and simple choice of $D'(s)$.

The simplest choice is $D'(s) = K$ for a constant K. We may use a number of methods to check the stability of the system for this choice, but a rough sketch of the root-locus, as shown below, is simple enough to see the location of the closed-loop poles.



As we observe from the root-locus diagram, there is no value of K that would result in a stable closed-loop system; mainly because the asymptote angles are $\theta_a = \pm 60°$, $180°$, and there are poles on the imaginary axis.

In order to have the asymptote intersection and the angles stay inside the left-half plane, we need to have zeros in $D'(s)$. Since we are placing two poles, we may have up to two zeros in $D'(s)$. With only one zero, the asymptote angles will be $\theta_a = \pm 90°$, and once we make sure that the asymptote intersection is on the left-half plane, we will have a stable system. For

$$D'(s) = K(s + a),$$

or

$$D(s)G(s) = K\frac{3(s+4)(s+a)}{s(s^2+4)(s+8)},$$

the asymptote intersection

$$\sigma_a = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{((-8) + (0) + (j2) + (-j2)) - ((-4) + (-a))}{4-2} = \frac{a-4}{2},$$

where $\sum p_i$ and $\sum z_i$ are the sums of the pole and zero locations, respectively. As long as $a < 4$, we get $\sigma_a < 0$. We also need to have the zero on the left-half plane, to have the pole at $s = 0$ stay stable.

Therefore, one possible simplest controller is

$$D(s) = K\frac{s+a}{s^2+4},$$

where $0 < a < 4$, and $K \gg 0$.

There is no need to do the general analysis in this problem, but we may get a tighter relationship from the Routh-Hurwitz's stability criterion on the characteristic polynomial. From the characteristic equation, $1 + D9s)G(s) = 0$, we have

$$1 + \frac{3K(s+4)(s+a)}{s(s^2+4)(s+8)} = 0,$$

or

$$s^4 + 8s^3 + \big(3K+4\big)s^2 + \big(3K(a+4)+32\big)s + 12Ka = 0.$$

The Routh-Hurwitz table for the system becomes as given below.

s^4	1	$3K+4$	$12Ka$
s^3	8	$3K(a+4)+32$	
s^2	$\frac{(8)(3K+4)-(1)(3K(a+4)+32)}{8} = \frac{3K(4-a)}{8}$	$12Ka$	
s	α		
1	$12Ka$		

Here, α is such that

$$\begin{aligned}\big(3K(4-a)\big)\alpha &= \big(3K(4-a)\big)\big(3K(a+4)+32\big) - 8(8)\,(12Ka)\\ &= 3K\big(3K(16-a^2) + (128-288a)\big).\end{aligned}$$

The 1-term gives

$$Ka > 0.$$

The s^2-term gives

$$3K(4-a) > 0,$$

or

$$a < 4 \qquad \text{and} \qquad K > 0,$$

since $a > 4$ and $K < 0$ would violate the 1-term condition. The s-term gives

$$K > 0 \qquad \text{and} \qquad K > \frac{288a - 128}{3(16-a^2)}.$$

Finding the intersection of all the conditions, we get

$$0 < a < 4 \qquad \text{and} \qquad K > \max\left\{0, \frac{288a-128}{3(16-a^2)}\right\}.$$

3. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{(s-2)(s+1)}{(s^2-4s+29)(s^2+6s+10)} = K\frac{s^2-s-2}{s^4+2s^3+15s^2+134s+290}.$$

(a) Determine the values of K such that the closed-loop system is asymptotically stable.

Solution: The stability of the closed-loop system can be determined using the Routh-Hurwitz's stability criterion on the characteristic polynomial. From the characteristic equation, $1+G(s)=0$, we have

$$1+K\frac{s^2-s-2}{s^4+2s^3+15s^2+134s+290}=0,$$

or

$$s^4+2s^3+(15+K)s^2+(134-K)s+(290-2K)=0.$$

The Routh-Hurwitz table for the system becomes as given below.

s^4	1	$15+K$	$290-2K$
s^3	2	$134-K$	
s^2	$\frac{(2)(15+K)-(1)(134-K)}{2}=\frac{3K-104}{2}$	$290-2K$	
s	α		
1	$290-2K$		

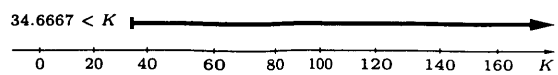
Here, α is such that

$$\begin{aligned}\alpha &= \frac{(3K-104)(134-K)-2(2)(290-2K)}{3K-104}\\ &= \frac{-3K^2+514K-15096}{3K-104}\\ &= \frac{-3(K-37.6377)(K-133.6956)}{3(K-34.6667)}.\end{aligned}$$

The Routh-Hurwitz's stability criterion implies the following conditions.

i. $3K-104>0$.

$$K>34.6667.$$



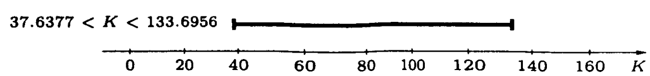
ii. $(-3(K-37.6377)(K-133.6956))/(3(K-34.6667))>0$.

A. $K-34.6667>0$ Case:

$$-3(K-37.6377)(K-133.6956)>0.$$
$$(K-37.6377)(K-133.6956)<0,$$

or

$$37.6377<K<133.6956.$$

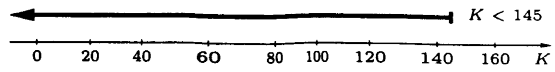


B. $K-34.6667<0$ Case:

This case results in instability from the previous condition.

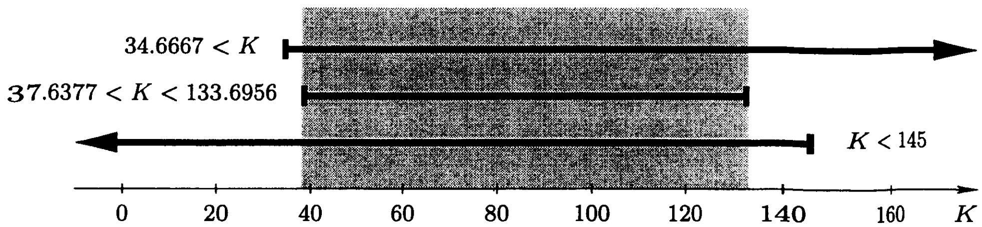
iii. $290 - 2K > 0$.

$$K < 145.$$



The intersection of all these regions leads to

$$37.6377 < K < 133.6956.$$



(b) Determine the value (or values) of K and the natural frequency (or frequencies), such that the closed-loop system would have sustained oscillations.

Solution: For sustained oscillations, we need to choose K, such that there are distinct poles on the imaginary axis and no pole on the right-half plane. The candidates for such a choice are obtained by generating a row of zeros on the Routh-Hurwitz table. Observing from the table, the only two such rows are the s and the 1-rows. However, the 1-row gives an imaginary-axis crossing at $s = 0$. Considering the elements on the s-row, we get

$$\frac{-3(K - 37.6377)(K - 133.6956)}{3(K - 34.6667)} = 0.$$

The solution of the above equation gives $K = 37.6377$ and $K = 133.6956$.

Next, we need to obtain the factors of the original polynomial from the previous row, and verify that we get poles on the imaginary axis. From the upper or the s^2-row,

$$\left((3K - 104)/2s^2 + (290 - 2K)\right)_{K=37.6377,133.6956} = 0.$$

Note here that the above equation gives some of the poles of the closed-loop system *only* for the values of K that make the s-row all zero.

For $K = 37.6377$, we get $s = \pm j6.9413$, and for $K = 133.6956$, we get $s = \pm j0.3901$. So for both of the cases, we have imaginary-axis crossings. And, from the first elements of the remaining rows of the Routh-Hurwitz table, we conclude that the rest of the poles are in the left-half plane. Therefore, the natural frequencies, such that the closed-loop system would have sustained oscillations, are $\omega_1 = 0.3901$ rad/s when $K = 133.6956$ and $\omega_2 = 6.9413$ rad/s when $K = 37.6377$.

4. Consider a negative unity-feedback control system with the open-loop transfer function

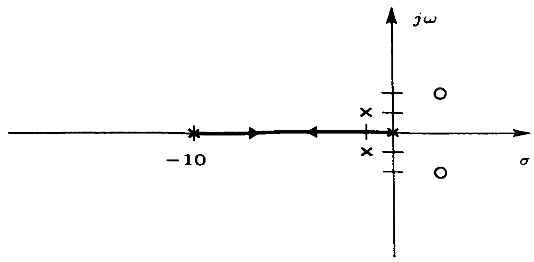
$$G(s) = K\frac{s^2-4s+8}{s(s^2+2s+2)(s+10)} = K\frac{s^2-4s+8}{s^4+12s^3+22s^2+20s}.$$

Construct the root-locus diagram. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angle of arrivals and/or departures.

Solution: First, we sketch the pole-zero locations and the real-axis portion of the root-locus diagram. Then, we decide the important features to be determined.

Need to determine:

- Asymptotes,
- Break-away point,
- Imaginary-axis crossings, and
- Angle of departures and arrivals.



Asymptotes

Real-Axis Crossing: $\sigma_a = \left(\sum p_i - \sum z_i\right)/(n-m)$

The real-axis crossing of the asymptotes is at

$$\sigma_a = \frac{\sum_i p_i - \sum_i z_i}{n-m} = \frac{((-10)+(-1+j1)+(-1-j1)+(0)) - ((2+j2)+(2-j2))}{4-2} = -8.$$

Real-Axis Angles: $\theta_a = \pm(2k+1)\pi/(n-m)$

The angles that the asymptotes make with the real axis are determined from

$$\theta_a = \frac{\pm(2k+1)\pi}{n-m} = \frac{\pm(2k+1)\pi}{4-2} = \pm\frac{\pi}{2}.$$

Break-Away Point: $\mathrm{d}K/\mathrm{d}s = 0$

From the characteristic equation,

$$1+G(s)=0,$$

$$1+K\frac{s^2-4s+8}{s^4+12s^3+22s^2+20s}=0,$$

and

$$-K = \frac{s^4+12s^3+22s^2+20s}{s^2-4s+8}.$$

Therefore,

$$-\frac{\mathrm{d}K}{\mathrm{d}s} = \frac{(4s^3+36s^2+44s+20)(s^2-4s+8)-(s^4+12s^3+22s^2+20s)(2s-4)}{(s^2-4s+8)^2}$$

$$= \frac{2(s^5-32s^3+90s^2+176s+80)}{(s^2-4s+8)^2}.$$

and for $dK/ds = 0$, the equation

$$s^5 - 32s^3 + 90s^2 + 176s + 80 = 0$$

gives $s = -6.48$, $s = -0.72 \pm j0.274$, and $s = 3.96 \pm j2.263$. So, the break-away point is at $s = -6.48$, since it is real and between -10 and 0.

Imaginary-Axis Crossings: Routh-Hurwitz Table

The imaginary axis crossings can be determined from the Routh-Hurwitz table. From the characteristic equation,

$$1 + G(s) = 0,$$

$$1 + K\frac{s^2 - 4s + 8}{s^4 + 12s^3 + 22s^2 + 20s} = 0,$$

$$s^4 + 12s^3 + 22s^2 + 20s + K(s^2 - 4s + 8) = 0,$$

we get the characteristic polynomial

$$q(s) = s^4 + 12s^3 + (22 + K)s^2 + (20 - 4K)s + 8K.$$

The Routh-Hurwitz table for this characteristic polynomial is given below.

s^4	1	$22 + K$	$8K$
s^3	12	$20 - 4K$	
s^2	$\frac{(12)(22+K) - (1)(20-4K)}{12} = \frac{244+16K}{12}$	$8K$	
s	α		
1	$8K$		

Here, α is such that

$$\begin{aligned}\alpha &= \frac{(244 + 16K)(20 - 4K) - 12(12)(8K)}{244 + 16K} \\ &= \frac{-64K^2 - 1808K + 4880}{244 + 16K} \\ &= \frac{-64(K + 30.7312)(K - 2.4812)}{244 + 16K}.\end{aligned}$$

The imaginary-axis crossings will correspond to the positive values of K that would make a row of all zeros on the table. The first such candidate is the s-row. The s-row is all zero, when $K = -30.7312$ or $K = 2.4812$. For the positive value of K, we get a factor of the characteristic polynomial from the upper or the s^2-row. So,

$$\left(\left(\frac{244 + 16K}{12}\right)s^2 + 8K\right)_{K=2.4812} = 0,$$

or $s = \pm j0.9163$. Therefore, the imaginary-axis crossings are at $s = \pm j0.9163$.

Angle of Departures and Arrivals: $\sum \measuredangle(\cdot) = \pm(2k+1)\pi$

The angles of departures from complex open-loop poles are determined from the angular conditions about the open-loop poles. Therefore, the angular condition about $s = -1 + j1$ is

$$-\measuredangle(s-(-10)) - \measuredangle(s-(-1-j1)) - \measuredangle(s-(-1+j1)) - \measuredangle(s-(0)) + \measuredangle(s-(2-j2)) + \measuredangle(s-(2+j2)) = 180^\circ + k360^\circ,$$

$$-\tan^{-1}\left(\frac{(1)-(0)}{(-1)-(-10)}\right) - \tan^{-1}\left(\frac{(1)-(-1)}{(-1)-(-1)}\right) - \theta_{\text{dep}} - \tan^{-1}\left(\frac{(1)-(0)}{(-1)-(0)}\right) + \tan^{-1}\left(\frac{(1)-(-2)}{(-1)-(2)}\right) + \tan^{-1}\left(\frac{(1)-(2)}{(-1)-(2)}\right) = 180^\circ + k360^\circ,$$

or

$$-6.34^\circ - 90^\circ - \theta_{\text{dep}} - 135^\circ + 135^\circ + 198.43^\circ = 180^\circ + k360^\circ.$$

As a result,

$$\theta_{\text{dep}} = -77.91^\circ.$$

The angles of arrivals to complex open-loop zeros are also determined from the angular conditions about the open-loop zeros. Therefore, the angular condition about $s = 2 + j2$ is

$$-\measuredangle(s-(-10)) - \measuredangle(s-(-1-j1)) - \measuredangle(s-(-1+j1)) - \measuredangle(s-(0)) + \measuredangle(s-(2-j2)) + \measuredangle(s-(2+j2)) = 180^\circ + k360^\circ,$$

$$-\tan^{-1}\left(\frac{(2)-(0)}{(2)-(-10)}\right) - \tan^{-1}\left(\frac{(2)-(-1)}{(2)-(-1)}\right) - \tan^{-1}\left(\frac{(2)-(1)}{(2)-(-1)}\right) - \tan^{-1}\left(\frac{(2)-(0)}{(2)-(0)}\right) + \tan^{-1}\left(\frac{(2)-(-2)}{(2)-(2)}\right) + \theta_{\text{arr}} = 180^\circ + k360^\circ,$$

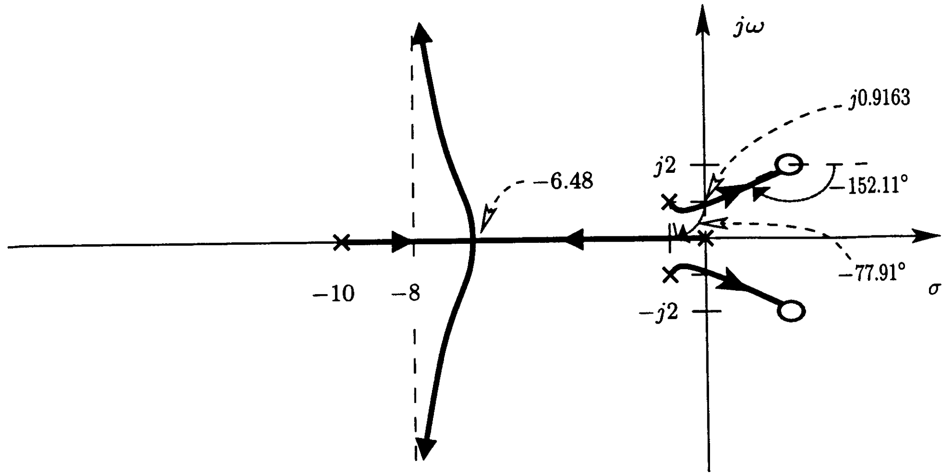
or

$$-9.46^\circ - 45^\circ - 18.43^\circ - 45^\circ + 90^\circ + \theta_{\text{arr}} = 180^\circ + k360^\circ.$$

As a result,

$$\theta_{\text{arr}} = -152.11^\circ.$$

With the features determined, we can now sketch the root-locus diagram.


jω
j0.9163
j2
−6.48
−152.11°
−77.91°
−10
−8
σ
−j2