EE 231 **Exam#2** May 01, 2007

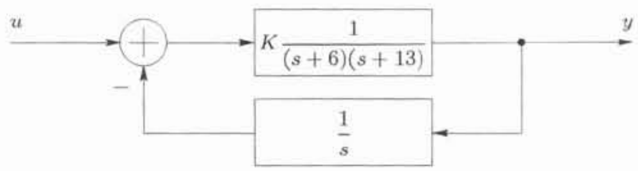
75 minutes

1. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = \frac{100}{s(0.1s+1)}.$$

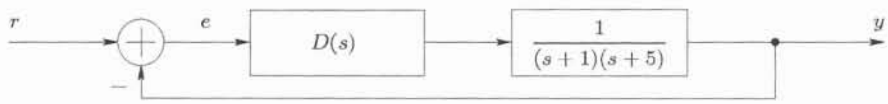
Determine the output steady-state error $e(\infty) = y(\infty) - r(\infty)$, where y is the output and r is the reference input, such that $r(t) = 1 + t + at^2$ for $t \geq 0$. (10pts)

2. Consider the following control system.



Determine the range of the constant K, such that the 2% settling-time is less than 4 seconds. (20pts)

3. Consider the following feedback control system.

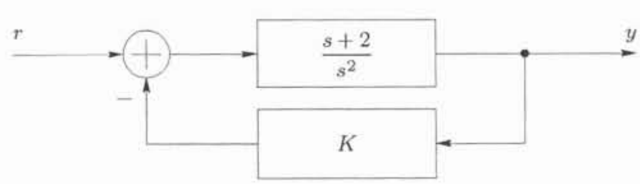


Consider a proportional-integral (PI) controller

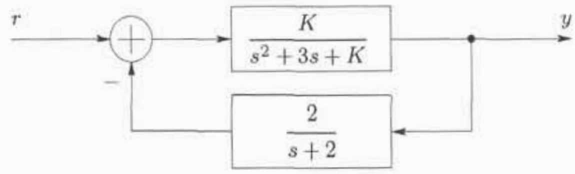
$$D(s) = K_P + \frac{K_I}{s} = \frac{K_P s + K_I}{s}.$$

Determine the constants K_P and K_I, such that the closed-loop system has sustained oscillations at a frequency of 2 rad/s. (20pts)

4. First sketch the root-locus diagram for the following control system. Then from the root-locus diagram, determine the value of K such that the closed-loop poles have a damping constant of $\sqrt{2}/2$. (20pts)



5. For the following feedback control system, construct the root-locus diagram for positive values of K. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angles of departure and/or arrival. (30pts)



EE 231 | Exam#2 Solutions | May 01, 2007

Exam#2
Solutions

1. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = \frac{100}{s(0.1s+1)}.$$

Determine the output steady-state error $e(\infty) = y(\infty) - r(\infty)$, where y is the output and r is the reference input, such that $r(t) = 1 + t + at^2$ for $t \geq 0$.

Solution: From the open-loop gain of the system, we determine that the system is of type-1. Therefore, the steady-state error

$$e(\infty) = \begin{cases} 0, & \text{if } r \text{ is a step input;} \\ 1/K_v, & \text{if } r \text{ is the unit-ramp input;} \\ \infty, & \text{if } r \text{ is a positive parabolic input.} \end{cases}$$

Here, K_v is the velocity steady-state error coefficient, where

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{100}{s(0.1s+1)} = 100.$$

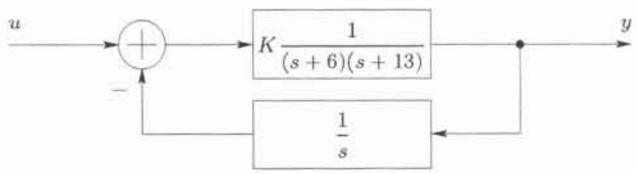
Since the system is linear; when $r(t) = 1 + t + at^2$, we get

$$e(\infty) = \begin{cases} 0 + 1/100 + \infty, & \text{if } a > 0; \\ 0 + 1/100, & \text{if } a = 0; \\ 0 + 1/100 - \infty, & \text{if } a < 0; \end{cases}$$

or

$$e(\infty) = \begin{cases} \infty, & \text{if } a > 0; \\ 0.01, & \text{if } a = 0; \\ -\infty, & \text{if } a < 0. \end{cases}$$

2. Consider the following control system.



Determine the range of the constant K, such that the 2% settling-time is less than 4 seconds.

Solution: Since the 2% settling time $t_{2\%s} = (4/\sigma_o)$; we have

$$t_{2\%s} = \frac{4}{\sigma_o} < 4,$$

$\sigma_o > 1$, or the poles need to be on the left-hand-side of the $\Re[s] = -\sigma_o = \sigma = -1$ vertical line. One way to determine the conditions for the poles to be on the left-hand-side of a vertical line is to use the Routh-Hurwitz's Table after shifting the vertical line from the $\sigma = 0$ line to the desired line.

The closed-loop poles are determined from the factors of the characteristic polynomial or the denominator of the closed-loop transfer function. In our case, the characteristic equation is

$$1 + G(s)H(s) = 1 + \left(K\frac{1}{(s+6)(s+13)}\right)\left(\frac{1}{s}\right) = \frac{s(s+6)(s+13)+K}{s(s+6)(s+13)} = 0,$$

and the characteristic polynomial becomes

$$q_c(s) = s(s+6)(s+13) + K.$$

If we use the Routh-Hurwitz's Table on this polynomial, we would determine the conditions for the poles to be on the left-hand-side of the $\sigma = 0$ vertical line. To determine the conditions for the left-hand-side of the $\sigma = -1$ line, we need to shift the characteristic polynomial, such that

$$\begin{aligned} q_c(s-1) &= (s-1)\big((s-1)+6\big)\big((s-1)+13\big) + K \\ &= (s-1)(s+5)(s+12) + K \\ &= s^3 + 16s^2 + 43s + (K-60). \end{aligned}$$

With the new polynomial, the Routh-Hurwitz's Table becomes as given below.

s^3	1	43
s^2	16	$K-60$
s	$\frac{(16)(43)-(1)(K-60)}{16}$	
1	$K-60$	

Applying the Routh-Hurwitz's criterion on the new polynomial gives the conditions for the left-hand-side of the $\sigma = -1$ line. The s-term gives

$$\frac{(16)(43)-(1)(K-60)}{16} > 0,$$

or

$$K < 748.$$

The 1-term gives

$$K - 60 > 0,$$

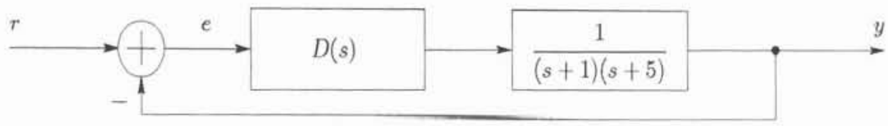
or

$$K > 60.$$

Therefore, from the intersection of the two regions, we get

$$60 < K < 748.$$

3. Consider the following feedback control system.



Consider a proportional-integral (PI) controller

$$D(s) = K_P + \frac{K_I}{s} = \frac{K_P s + K_I}{s}.$$

Determine the constants K_P and K_I, such that the closed-loop system has sustained oscillations at a frequency of 2 rad/s.

Solution: For sustained oscillations, we need to choose the constants, such that there are distinct poles on the imaginary axis and no pole on the right-half plane. The candidates for such a choice are obtained by generating a row of zeros on the Routh-Hurwitz's Table.

From the characteristic equation, $1 + D(s)G(s) = 0$, we have

$$1 + \left(\frac{K_P s + K_I}{s}\right)\left(\frac{1}{(s+1)(s+5)}\right) = 0,$$

or

$$s^3 + 6s^2 + (5 + K_P)s + K_I = 0.$$

The Routh-Hurwitz's Table for the system becomes as given below.

s^3	1	$5 + K_P$
s^2	6	K_I
s	$\frac{(6)(5+K_P) - (1)(K_I)}{6}$	
1	K_I	

Observing from the table, the only possible all-zero rows are the s and the 1-rows. However, the 1-row gives an imaginary-axis crossing at $s = 0$. Considering the elements on the s-row, we get

$$\frac{(6)(5+K_P) - (1)(K_I)}{6} = 0.$$

The solution of the above equation gives

$$6K_P - K_I = -30.$$

For such a choice of constants, the upper or the s^2-row gives

$$6s^2 + K_I = 0,$$

and the solution of this equation $s = \pm j\sqrt{K_I/6}$ gives the imaginary axis crossings or the frequency of oscillation. Setting

$$s = \pm j\sqrt{K_I/6} = \pm j\omega_n = \pm j2,$$

we get $\sqrt{K_I/6} = 2$ or $K_I = 24$. Since we also need to have $6K_P - K_I = -30$, we get $K_P = -1$.

To ensure stability of the system, we also need to check the location of the remaining pole. The poles of the closed-loop system can be determined from the characteristic equation.

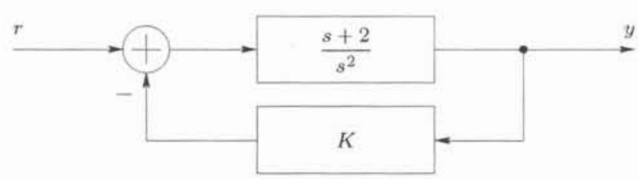
$$1 + D(s)G(s) = 1 + \left(\frac{-s+24}{s}\right)\left(\frac{1}{(s+1)(s+5)}\right) = \frac{s(s+1)(s+5) + (-s+24)}{s(s+1)(s+5)}$$

$$= \frac{s^3 + 6s^2 + 4s + 24}{s(s+1)(s+5)} = \frac{(s^2+4)(s+6)}{s(s+1)(s+5)} = 0.$$

Since the remaining pole turns out to be at $s = -6$, the closed-loop system will have sustained oscillations at a frequency of 2 rad/s, when the control is

$$D(s) = -1 + \frac{24}{s}.$$

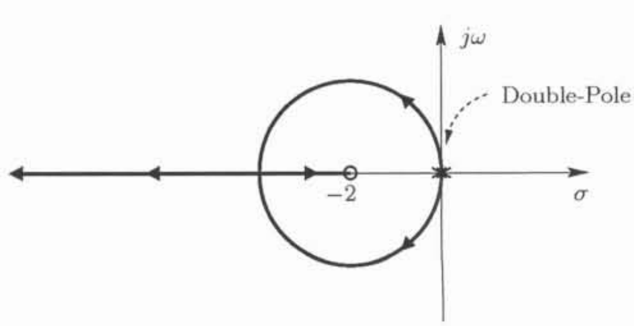
4. First sketch the root-locus diagram for the following control system. Then from the root-locus diagram, determine the value of K such that the closed-loop poles have a damping constant of $\sqrt{2}/2$.



Solution: From the open-loop gain of the system

$$KG(s) = K\frac{s+2}{s^2},$$

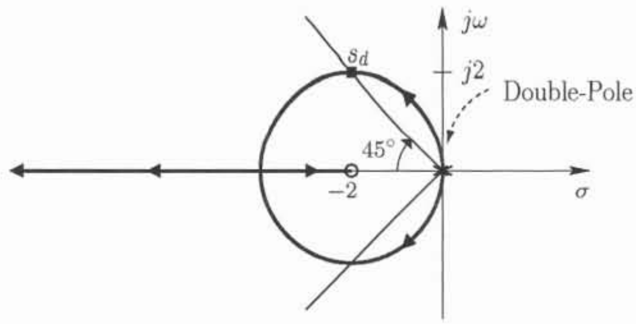
we realize that there are two poles and one zero. As a result, the root-locus diagram will consist of a circle centered at the zero and with a radius equal to the geometric mean of the distances from the poles to the zero.



In order for the closed-loop poles to have a damping constant of $\sqrt{2}/2$, they need to be on the radial line with an angle of $\cos^{-1}(\zeta)$ with respect to the negative real axis. In our case, that angle

$$\alpha = \cos^{-1}(\zeta) = \cos^{-1}(\sqrt{2}/2) = 45^\circ.$$

Plotting this radial on the root-locus diagram gives us the location of the desired closed-loop poles.

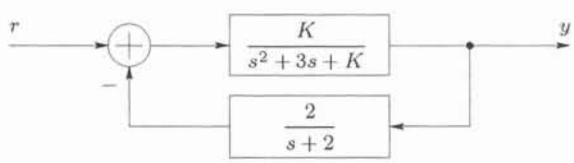


We observe that the radial line and the root-locus diagram intersect. The intersection point, $s_d = -2 + j2$, satisfies the damping-constant requirement, and it is also on the root-locus diagram. As a result, we can determine the constant K from the magnitude condition.

$$\left|KG(s)\right|_{s=s_d} = \left|K\frac{s+2}{s^2}\right|_{s=-2+j2} = 1,$$

or $K = 4$. Therefore, the closed-loop poles will have a damping constant of $\sqrt{2}/2$, when $K = 4$.

5. For the following feedback control system, construct the root-locus diagram for positive values of K. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angles of departure and/or arrival.



Solution: The sketch of the location of the closed-loop poles is the root-locus diagram. However, in this case the open-loop gain of the system is

$$G(s)H(s) = \left(\frac{K}{s^2+3s+K}\right)\left(\frac{2}{s+2}\right) = \frac{2K}{(s+2)(s^2+3s+K)},$$

where the root-locus variable K is not a multiplicative coefficient of the open-loop gain. So, we need to convert the problem into the conventional form while preserving the location of the closed-loop poles the same. The closed-loop poles are obtained from the characteristic equation, where

$$1 + G(s)H(s) = 0,$$

or

$$1 + \frac{2K}{(s+2)(s^2+3s+K)} = 0,$$

$$\frac{(s+2)(s^2+3s+K)+2K}{(s+2)(s^2+3s+K)} = 0,$$

$$(s+2)(s^2+3s+K)+2K = 0,$$

$$s^3+5s^2+6s+Ks+4K = 0.$$

We need to regroup the characteristic equation, so that the characteristic equation is in the form

$$1 + K\frac{n(s)}{d(s)} = 0,$$

for some polynomials $n(s)$ and $d(s)$. So,

$$s^3+5s^2+6s+Ks+4K = 0,$$

$$(s^3+5s^2+6s)+K(s+4) = 0,$$

$$\frac{(s^3+5s^2+6s)+K(s+4)}{(s^3+5s^2+6s)} = 0,$$

$$1 + K\frac{s+4}{s^3+5s^2+6s} = 0.$$

Therefore, the new open-loop gain

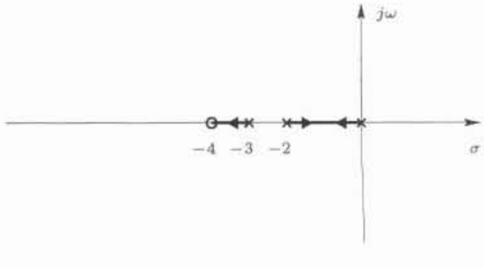
$$G'(s)H'(s) = K\frac{s+4}{s^3+5s^2+6s} = K\frac{s+4}{s(s+2)(s+3)}$$

generates the same closed-loop poles as the original open-loop gain, but the open-loop gain $G'(s)H'(s)$ of the new system is in the usual form for the generation of the root-locus diagram. In other words, the locations of the closed-loop poles based on the open-loop gains $G(s)H(s)$ and $G'(s)H'(s)$ are identical, however we can use the regular root-locus drawing techniques on the primed system.

First, we sketch the pole-zero locations and the real-axis portion of the root-locus diagram. Then, we decide the important features to be determined.

Need to determine:

- Asymptotes, and
- Break-away point.



We may also need to determine the imaginary-axis crossings depending on the orientation of the asymptotes.

Asymptotes

Real-Axis Crossing: $\sigma_a = \left(\sum p_i - \sum z_i\right)/(n-m)$

The real-axis crossing of the asymptotes is at

$$\sigma_a = \frac{\sum_i p_i - \sum_i z_i}{n-m} = \frac{\left((0)+(-2)+(-3)\right)-(-4)}{3-1} = \frac{-1}{2} = -0.5.$$

Real-Axis Angles: $\theta_a = \pm(2k+1)\pi/(n-m)$

The angles that the asymptotes make with the real axis are determined from

$$\theta_a = \frac{\pm(2k+1)\pi}{n-m} = \frac{\pm(2k+1)\pi}{3-1} = \pm\frac{\pi}{2}.$$

Break-Away Point: $\mathrm{d}K/\mathrm{d}s = 0$

From the characteristic equation,

$$1 + G'(s)H'(s) = 0,$$

$$1 + K\frac{s+4}{s^3+5s^2+6s} = 0,$$

and

$$-K = \frac{s^3+5s^2+6s}{s+4}.$$

Note that this is the same equation for the original system as well. Therefore,

$$-\frac{\mathrm{d}K}{\mathrm{d}s} = \frac{(3s^2+10s+6)(s+4)-(s^3+5s^2+6s)(1)}{(s+4)^2}. = \frac{2s^3+17s^2+40s+24}{(s+4)^2}.$$

and for $\mathrm{d}K/\mathrm{d}s = 0$, the equation

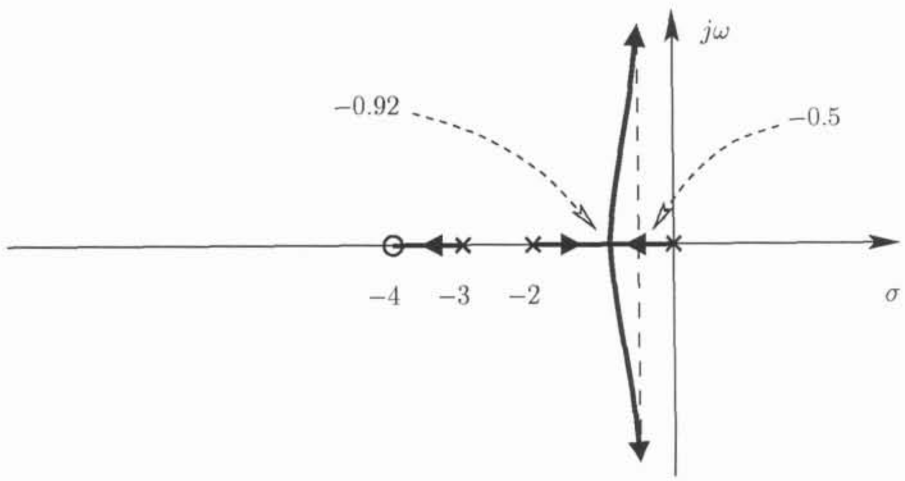
$$2s^3+17s^2+40s+24 = 0$$

gives

$$s = -4.9483, -2.6294, -0.9223.$$

The break-away point for positive K is the solution between -2 and 0 that is $s = -0.92$.

From the asymptote angles, we realize that there will not be any imaginary-axis crossings. With the features determined, we can now sketch the root-locus diagram.



jω
−0.92
−0.5
−4
−3
−2
σ