EE 265 **Exam#1** **Feb. 14, 2007**

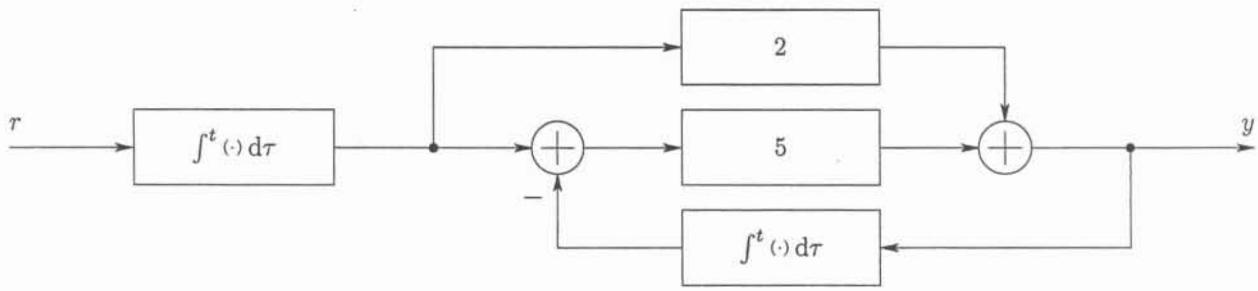
50 minutes

1. Given the signal $x(t)$, determine $y(t) = 4x\big(2(5-3t)\big)$ in the given form of the signal.

$$x(t) = \begin{cases} 0, & \text{if } t \le 0; \\ \sin(3t), & \text{if } 0 < t < 2\pi; \\ 3t - 6\pi, & \text{if } 2\pi \le t < 6\pi; \\ 12\pi, & \text{if } 6\pi \le t. \end{cases}$$

(15pts)

2. Determine the system equation relating the output y to the input r for the following block diagram. The final expression should contain derivative terms and no integral terms. (20pts)



3. Consider the function x, where

$$x(t) = \begin{cases} 0, & \text{if } -2 \le t \le -1; \\ t, & \text{if } -1 < t < 1; \\ 0, & \text{if } 1 \le t \le 2. \end{cases}$$

Determine the trigonometric fourier-series expansion of x for $-2 \le t \le 2$. Simplify the expression as much as possible. (25pts)

HINT: Some of the indefinite integrals below might be helpful.

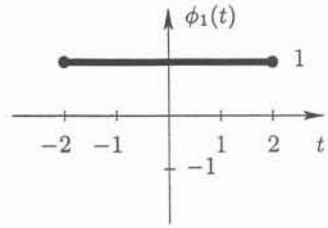
$$\int^t \sin(\alpha\tau)\,\mathrm{d}\tau = -\left(\frac{1}{\alpha}\right)\cos(\alpha t). \qquad \int^t \cos(\alpha\tau)\,\mathrm{d}\tau = \left(\frac{1}{\alpha}\right)\sin(\alpha t).$$

$$\int^t \tau\sin(\alpha\tau)\,\mathrm{d}\tau = \left(\frac{1}{\alpha^2}\right)\sin(\alpha t) - \left(\frac{t}{\alpha}\right)\cos(\alpha t). \qquad \int^t \tau\cos(\alpha\tau)\,\mathrm{d}\tau = \left(\frac{1}{\alpha^2}\right)\cos(\alpha t) + \left(\frac{t}{\alpha}\right)\sin(\alpha t).$$

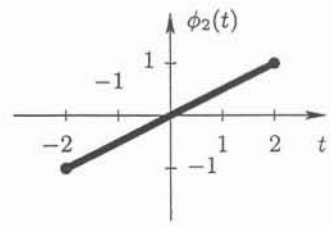
$$\int^t \tau^2\sin(\alpha\tau)\,\mathrm{d}\tau = \left(\frac{2t}{\alpha^2}\right)\sin(\alpha t) + \left(\frac{2}{\alpha^3} - \frac{t^2}{\alpha}\right)\cos(\alpha t). \qquad \int^t \tau^2\cos(\alpha\tau)\,\mathrm{d}\tau = \left(\frac{2t}{\alpha^2}\right)\cos(\alpha t) + \left(\frac{t^2}{\alpha} - \frac{2}{\alpha^3}\right)\sin(\alpha t).$$

4. Consider the following basis functions, ϕ_1, ϕ_2, and ϕ_3.

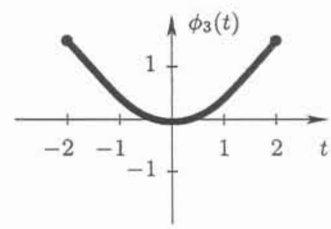
$$\phi_1(t) = 1 \text{ for } -2 \leq t \leq 2.$$



$$\phi_2(t) = (1/2)t \text{ for } -2 \leq t \leq 2.$$



$$\phi_3(t) = (3/8)t^2 \text{ for } -2 \leq t \leq 2.$$



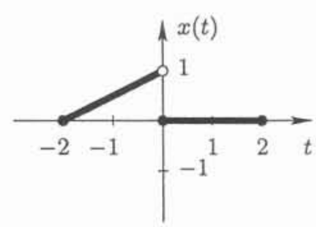
(a) Check whether or not the basis functions ϕ_i for $i = 1, 2, 3$ are orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-2}^{2} f(t)g(t)\,\mathrm{d}t.$$

If they are not orthogonal, obtain an orthogonal set of basis functions that spans the same subspace. (15pts)

(b) Find the best representation of the function x, where

$$x(t) = \begin{cases} (1/2)t + 1, & \text{if } -2 \leq t < 0; \\ 0, & \text{if } 0 \leq t \leq 2. \end{cases}$$



as a linear combination of the orthogonal basis functions, such that the error based on the associated norm square, $\|(\cdot)\|^2 = \int_{-2}^{2} (\cdot)^2\,\mathrm{d}t$, is minimized. (25pts)

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Solutions

1. Given the signal $x(t)$, determine $y(t) = 4x\big(2(5-3t)\big)$ in the given form of the signal.

$$x(t) = \begin{cases} 0, & \text{if } t \le 0; \\ \sin(3t), & \text{if } 0 < t < 2\pi; \\ 3t - 6\pi, & \text{if } 2\pi \le t < 6\pi; \\ 12\pi, & \text{if } 6\pi \le t. \end{cases}$$

Solution: One approach is to realize that the desired function is $y(\tau) = 4x(2(5-3\tau)) = 4x(10-6\tau)$.

$$4x(10-6\tau) = \begin{cases} 4\times 0, & \text{if } (10-6\tau) \le 0; \\ 4\times \sin\big((3(10-6\tau)\big), & \text{if } 0 < (10-6\tau) < 2\pi; \\ 4\times \big(3(10-6\tau)-6\pi\big), & \text{if } 2\pi \le (10-6\tau) < 6\pi; \\ 4\times 12\pi, & \text{if } 6\pi \le (10-6\tau), \end{cases}$$

$$= \begin{cases} 0, & \text{if } -6\tau \le -10; \\ 4\sin\big(30-18\tau\big), & \text{if } -10 < -6\tau < 2\pi - 10; \\ 120 - 24\pi - 72\tau, & \text{if } 2\pi - 10 \le -6\tau < 6\pi - 10; \\ 48\pi, & \text{if } 6\pi - 10 \le -6\tau, \end{cases}$$

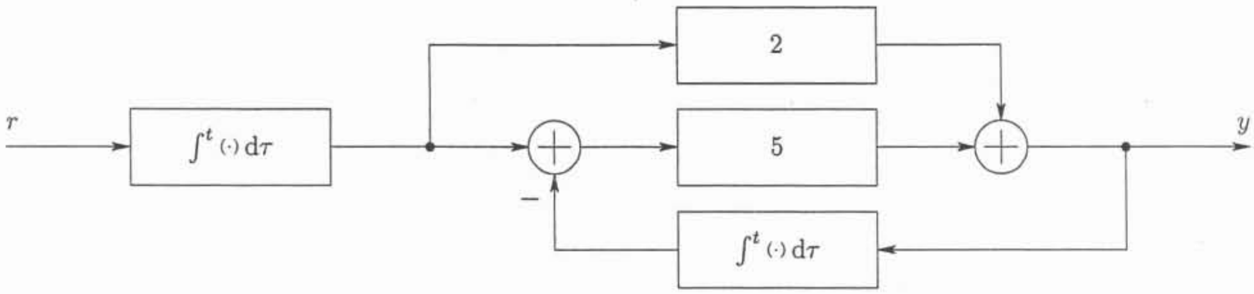
$$= \begin{cases} 0, & \text{if } -\tau \le -5/3; \\ 4\sin\big(30-18\tau\big), & \text{if } -5/3 < -\tau < \pi/3 - 5/3; \\ 120 - 24\pi - 72\tau, & \text{if } \pi/3 - 5/3 \le -\tau < \pi - 5/3; \\ 48\pi, & \text{if } \pi - 5/3 \le -\tau, \end{cases}$$

$$= \begin{cases} 0, & \text{if } \tau \ge 5/3; \\ 4\sin\big(30-18\tau\big), & \text{if } 5/3 > \tau > -\pi/3 + 5/3; \\ 120 - 24\pi - 72\tau, & \text{if } -\pi/3 + 5/3 \ge \tau > -\pi + 5/3; \\ 48\pi, & \text{if } -\pi + 5/3 \ge \tau. \end{cases}$$

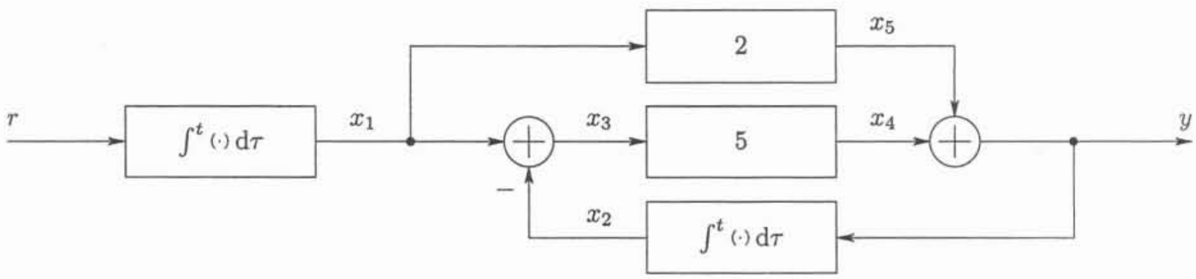
After rearranging the terms in increasing order, we get

$$y(\tau) = 4x(2(5-3\tau)) = \begin{cases} 48\pi, & \text{if } \tau \le -\pi + 5/3; \\ 120 - 24\pi - 72\tau, & \text{if } -\pi + 5/3 < \tau \le -\pi/3 + 5/3; \\ 4\sin\big(30-18\tau\big), & \text{if } -\pi/3 + 5/3 < \tau < 5/3; \\ 0, & \text{if } 5/3 \le \tau. \end{cases}$$

2. Determine the system equation relating the output y to the input r for the following block diagram. The final expression should contain derivative terms and no integral terms.



Solution: We can proceed step by step tracing the input and the output signals on the block diagram.



Marking some of the signals as shown above, we have

$$
\begin{aligned}
x_1(t) &= \int^t r(\tau)\,\mathrm{d}\tau; \\
x_2(t) &= \int^t y(\tau)\,\mathrm{d}\tau; \\
x_3(t) &= x_1(t) - x_2(t) = \int^t r(\tau)\,\mathrm{d}\tau - \int^t y(\tau)\,\mathrm{d}\tau; \\
x_4(t) &= 5x_3(t) = 5\int^t r(\tau)\,\mathrm{d}\tau - 5\int^t y(\tau)\,\mathrm{d}\tau; \\
x_5(t) &= 2x_1(t) = 2\int^t r(\tau)\,\mathrm{d}\tau; \\
y(t) &= x_4(t) + x_5(t) = 5\int^t r(\tau)\,\mathrm{d}\tau - 5\int^t y(\tau)\,\mathrm{d}\tau + 2\int^t r(\tau)\,\mathrm{d}\tau = 7\int^t r(\tau)\,\mathrm{d}\tau - 5\int^t y(\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

The last equation is the system equation, since it has only the input and the output variables. However, we need to express it in terms of derivatives instead of integrals. Taking the derivative of the last equation, we get

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = 7r(t) - 5y(t),$$

or

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 5y(t) = 7r(t).$$

3. Consider the function x, where

$$x(t) = \begin{cases} 0, & \text{if } -2 \le t \le -1; \\ t, & \text{if } -1 < t < 1; \\ 0, & \text{if } 1 \le t \le 2. \end{cases}$$

Determine the trigonometric fourier-series expansion of x for $-2 \le t \le 2$. Simplify the expression as much as possible.

HINT: Some of the indefinite integrals below might be helpful.

$$\int^t \sin(\alpha\tau)\,d\tau = -\left(\frac{1}{\alpha}\right)\cos(\alpha t). \qquad \int^t \cos(\alpha\tau)\,d\tau = \left(\frac{1}{\alpha}\right)\sin(\alpha t).$$

$$\int^t \tau\sin(\alpha\tau)\,d\tau = \left(\frac{1}{\alpha^2}\right)\sin(\alpha t) - \left(\frac{t}{\alpha}\right)\cos(\alpha t). \qquad \int^t \tau\cos(\alpha\tau)\,d\tau = \left(\frac{1}{\alpha^2}\right)\cos(\alpha t) + \left(\frac{t}{\alpha}\right)\sin(\alpha t).$$

$$\int^t \tau^2\sin(\alpha\tau)\,d\tau = \left(\frac{2t}{\alpha^2}\right)\sin(\alpha t) + \left(\frac{2}{\alpha^3} - \frac{t^2}{\alpha}\right)\cos(\alpha t). \qquad \int^t \tau^2\cos(\alpha\tau)\,d\tau = \left(\frac{2t}{\alpha^2}\right)\cos(\alpha t) + \left(\frac{t^2}{\alpha} - \frac{2}{\alpha^3}\right)\sin(\alpha t).$$

Solution: The trigonometric fourier-series of x is in the form of an infinite sum, such that

$$x(t) = a_0 + \sum_{n=1}^{\infty}\big(a_n\cos(n\omega t) + b_n\sin(n\omega t)\big),$$

where

$$a_0 = \frac{\langle x\,,1\rangle}{\langle 1\,,1\rangle} = \frac{1}{T}\int_T x(t)\,dt,$$

$$a_n = \frac{\langle x\,,\,\cos(n\omega(\cdot))\rangle}{\langle\cos(n\omega(\cdot))\,,\,\cos(n\omega(\cdot))\rangle} = \frac{2}{T}\int_T x(t)\cos(n\omega t)\,dt,$$

$$b_n = \frac{\langle x\,,\,\sin(n\omega(\cdot))\rangle}{\langle\sin(n\omega(\cdot))\,,\,\sin(n\omega(\cdot))\rangle} = \frac{2}{T}\int_T x(t)\sin(n\omega t)\,dt$$

for $n \ge 1$, and

$$\langle f\,,g\rangle = \int_T f(t)g(t)\,dt.$$

In our case $T = 4$ and $\omega = 2\pi/T = \pi/2$, so

$$a_0 = \frac{1}{4}\int_{-2}^{2} x(t)\,dt = \frac{1}{4}\int_{-1}^{1} t\,dt = \frac{1}{4}\left[\frac{t^2}{2}\right]_{t=-1}^{t=1} = \frac{1}{4}\left[\left(\frac{1^2}{2}\right) - \left(\frac{(-1)^2}{2}\right)\right] = 0,$$

$$\begin{aligned} a_n &= \frac{2}{4}\int_{-2}^{2} x(t)\cos(n\omega t)\,dt = \frac{1}{2}\int_{-1}^{1} t\cos\left(\frac{n\pi}{2}t\right)\,dt \\ &= \frac{1}{2}\left[\frac{1}{(n\pi/2)^2}\cos\left(\frac{n\pi}{2}t\right) + \frac{t}{n\pi/2}\sin\left(\frac{n\pi}{2}t\right)\right]_{t=-1}^{t=1} \\ &= \frac{1}{2}\left[\left(\frac{1}{(n\pi/2)^2}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n\pi/2}\sin\left(\frac{n\pi}{2}\right)\right) - \left(\frac{1}{(n\pi/2)^2}\cos\left(\frac{n\pi}{2}\right) - \frac{-1}{n\pi/2}\sin\left(\frac{n\pi}{2}\right)\right)\right] \\ &= 0, \end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{2}{4}\int_{-2}^{2} x(t)\sin(n\omega t)\,\mathrm{d}t = \frac{1}{2}\int_{-1}^{1} t\sin\left(\frac{n\pi}{2}t\right)\,\mathrm{d}t \\
&= \frac{1}{2}\left[\frac{1}{(n\pi/2)^2}\sin\left(\frac{n\pi}{2}t\right) - \frac{t}{n\pi/2}\cos\left(\frac{n\pi}{2}t\right)\right]_{t=-1}^{t=1} \\
&= \frac{1}{2}\left[\left(\frac{1}{(n\pi/2)^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi/2}\cos\left(\frac{n\pi}{2}\right)\right) - \left(-\frac{1}{(n\pi/2)^2}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n\pi/2}\cos\left(\frac{n\pi}{2}\right)\right)\right] \\
&= \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right).
\end{aligned}
$$

So,

$$x(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2}t\right),$$

where

$$b_n = \begin{cases} 4/(n\pi)^2, & \text{if } n = 1,\,5,\,9,\,\ldots; \\ 2/(n\pi), & \text{if } n = 2,\,6,\,10,\,\ldots; \\ -4/(n\pi)^2, & \text{if } n = 3,\,7,\,11,\,\ldots; \\ -2/(n\pi), & \text{if } n = 4,\,8,\,12,\,\ldots. \end{cases}$$

We may also rewrite the summation separating odd and even terms, such that

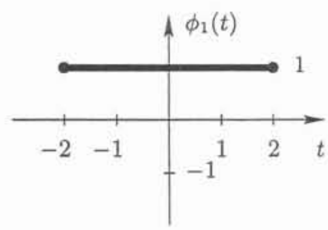
$$x(t) = \sum_{k=0}^{\infty}\left(\frac{(-1)^k 4}{(2k+1)^2\pi^2}\sin\left(\frac{(2k+1)}{2}\pi t\right) + \frac{(-1)^k 2}{(2k+2)\pi}\sin\left(\frac{(2k+2)}{2}\pi t\right)\right),$$

or

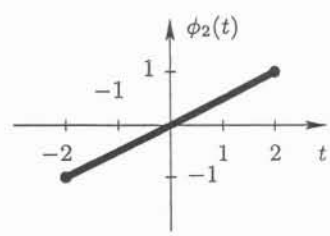
$$x(t) = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{(k+1/2)^2\pi^2}\sin\big((k+1/2)\pi t\big) + \frac{(-1)^k}{(k+1)\pi}\sin\big((k+1)\pi t\big)\right) \text{ for } -2 \le t \le 2.$$

4. Consider the following basis functions, ϕ_1, ϕ_2, and ϕ_3.

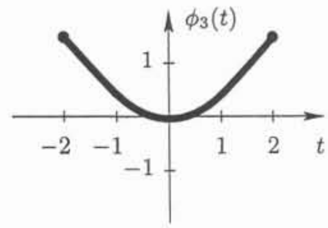
$\phi_1(t) = 1$ for $-2 \le t \le 2$.



$\phi_2(t) = (1/2)t$ for $-2 \le t \le 2$.



$\phi_3(t) = (3/8)t^2$ for $-2 \le t \le 2$.



(a) Check whether or not the basis functions ϕ_i for $i = 1, 2, 3$ are orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-2}^{2} f(t)g(t)\,\mathrm{d}t.$$

If they are not orthogonal, obtain an orthogonal set of basis functions that spans the same subspace.

Solution: To check orthogonality, we evaluate the inner products.

$$\langle \phi_1, \phi_2 \rangle = \int_{-2}^{2} \phi_1(t)\phi_2(t)\,\mathrm{d}t = \int_{-2}^{2} (1)\left((1/2)t\right)\mathrm{d}t = \frac{1}{2}\left[\frac{t^2}{2}\right]_{t=-2}^{t=2} = 0.$$

$$\langle \phi_1, \phi_3 \rangle = \int_{-2}^{2} \phi_1(t)\phi_3(t)\,\mathrm{d}t = \int_{-2}^{2} (1)\left((3/8)t^2\right)\mathrm{d}t = \frac{3}{8}\left[\frac{t^3}{3}\right]_{t=-2}^{t=2} = \frac{3}{8}\left[\frac{(2)^3}{3} - \frac{(-2)^3}{3}\right] = 2.$$

$$\langle \phi_2, \phi_3 \rangle = \int_{-2}^{2} \phi_2(t)\phi_3(t)\,\mathrm{d}t = \int_{-2}^{2} \left((1/2)t\right)\left((3/8)t^2\right)\mathrm{d}t = \frac{3}{16}\left[\frac{t^4}{4}\right]_{t=-2}^{t=2} = 0.$$

Therefore, ϕ_2 is orthogonal to both ϕ_1 and ϕ_3, but ϕ_1 and ϕ_3 are not orthogonal to each other.

To orthogonalize the set, we use the Gram-Schmidt's orthogonalization procedure. In this case, $\phi_1' = \phi_1$ and $\phi_2' = \phi_2$, since ϕ_1 and ϕ_2 are already orthogonal to each other. Here, $(\cdot)'$ represents the orthogonal elements.

$$\begin{aligned}
\phi_3'(t) &= \phi_3(t) - \frac{\langle \phi_3, \phi_2' \rangle}{\langle \phi_2', \phi_2' \rangle}\phi_2'(t) - \frac{\langle \phi_3, \phi_1' \rangle}{\langle \phi_1', \phi_1' \rangle}\phi_1'(t) \\
&= (3/8)t^2 - \frac{0}{\langle \phi_2', \phi_2' \rangle}\phi_2'(t) - \frac{2}{\int_{-2}^{2}(1)(1)\,\mathrm{d}t}1 \\
&= (3/8)t^2 - (1/2).
\end{aligned}$$

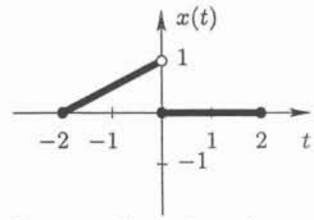
So, the new set of orthogonal functions are

$$\begin{aligned}
\phi_1'(t) &= 1, \\
\phi_2'(t) &= (1/2)t, \\
\phi_3'(t) &= (3/8)t^2 - (1/2)
\end{aligned}$$

for $-2 \le t \le 2$.

(b) Find the best representation of the function x, where

$$x(t) = \begin{cases} (1/2)t + 1, & \text{if } -2 \le t < 0; \\ 0, & \text{if } 0 \le t \le 2. \end{cases}$$



as a linear combination of the orthogonal basis functions, such that the error based on the associated norm square, $\|(\cdot)\|^2 = \int_{-2}^{2} (\cdot)^2 \, dt$, is minimized.

Solution: The minimal error in the associated-norm square sense is obtained when

$$x(t) \approx \sum_i k_i \phi_i(t),$$

where k_i is the ith generalized fourier-series coefficient. In our case,

$$x(t) \approx \sum_{i=1}^{3} k_i \phi_i'(t),$$

where

$$k_i = \frac{\langle x \, , \phi_i' \rangle}{\langle \phi_i' \, , \phi_i' \rangle}.$$

$$k_1 = \frac{\langle x \, , \phi_1' \rangle}{\langle \phi_1' \, , \phi_1' \rangle} = \frac{\int_{-2}^{0} \big((1/2)t + 1\big)\big(1\big) \, dt}{4} = \frac{\big[(1/4)t^2 + t\big]_{t=-2}^{t=0}}{4}$$

$$= \frac{\big[\big((1/4)(0)^2 + (0)\big) - \big((1/4)(-2)^2 + (-2)\big)\big]}{4} = \frac{1}{4}.$$

$$k_2 = \frac{\langle x \, , \phi_2' \rangle}{\langle \phi_2' \, , \phi_2' \rangle} = \frac{\int_{-2}^{0} \big((1/2)t + 1\big)\big((1/2)t\big) \, dt}{\int_{-2}^{2} \big((1/2)t\big)\big((1/2)t\big) \, dt} = \frac{\big[(1/12)t^3 + (1/4)t^2\big]_{t=-2}^{t=0}}{[(1/12)t^3]_{t=-2}^{t=2}}$$

$$= \frac{\big[\big((1/12)(0)^3 + (1/4)(0)^2\big) - \big((1/12)(-2)^3 + (1/4)(-2)^2\big)\big]}{\big[\big((1/12)(2)^3\big) - \big((1/12)(-2)^3\big)\big]} = \frac{-1/3}{4/3} = -\frac{1}{4}.$$

$$k_3 = \frac{\langle x \, , \phi_3' \rangle}{\langle \phi_3' \, , \phi_3' \rangle} = \frac{\int_{-2}^{0} \big((1/2)t + 1\big)\big((3/8)t^2 - (1/2)\big) \, dt}{\int_{-2}^{2} \big((3/8)t^2 - (1/2)\big)\big((3/8)t^2 - (1/2)\big) \, dt}$$

$$= \frac{\big[(3/64)t^4 + (1/8)t^3 - (1/8)t^2 - (1/2)t\big]_{t=-2}^{t=0}}{[(9/320)t^5 - (1/8)t^3 + (1/4)t]_{t=-2}^{t=2}}$$

$$= \frac{\left[\begin{array}{l} \big((3/64)(0)^4 + (1/8)(0)^3 - (1/8)(0)^2 - (1/2)(0)\big) \\ \qquad - \big((3/64)(-2)^4 + (1/8)(-2)^3 - (1/8)(-2)^2 - (1/2)(-2)\big) \end{array}\right]}{\left[\begin{array}{l} \big((9/320)(2)^5 - (1/8)(2)^3 + (1/4)(2)\big) \\ \qquad - \big((9/320)(-2)^5 - (1/8)(-2)^3 + (1/4)(-2)\big) \end{array}\right]}$$

$$= \frac{-1/4}{4/5} = -\frac{5}{16}.$$

Therefore,

$$x(t) \approx \left(\frac{1}{4}\right)\left(1\right) - \left(\frac{1}{4}\right)\left((1/2)t\right) - \left(\frac{5}{16}\right)\left((3/8)t^2 - (1/2)\right) \text{ for } -2 \leq t \leq 2.$$