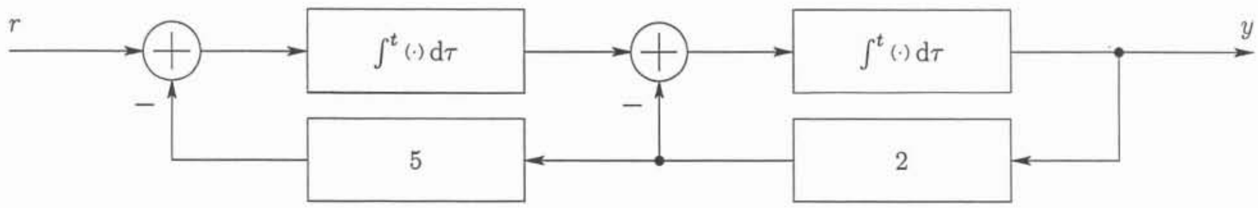
EE 261 | **Exam#1** | **Feb. 25, 1998**

50 minutes

1. Determine the system equation relating the output y to the input r for the following block diagram. The final expression should contain derivative terms and no integral terms. (20pts)



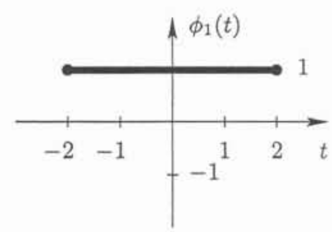
2. Consider the unit-pulse function p, where

$$p(t) = \begin{cases} 0, & \text{if } -2 \le t \le -1; \\ 1, & \text{if } -1 < t < 1; \\ 0, & \text{if } 1 \le t \le 2. \end{cases}$$

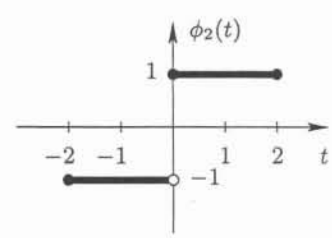
Determine the trigonometric fourier series expansion of p for $-2 \le t \le 2$. Simplify the expression as much as possible. (25pts)

3. Consider the following functions, ϕ_1, ϕ_2, and ϕ_3.

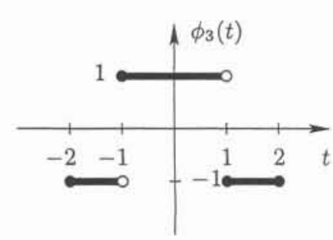
$$\phi_1(t) = 1 \text{ for } -2 \le t \le 2.$$



$$\phi_2(t) = \begin{cases} -1, & \text{if } -2 \le t < 0; \\ 1, & \text{if } 0 \le t \le 2. \end{cases}$$



$$\phi_3(t) = \begin{cases} -1, & \text{if } -2 \le t < -1; \\ 1, & \text{if } -1 \le t < 1; \\ -1, & \text{if } 1 \le t \le 2. \end{cases}$$



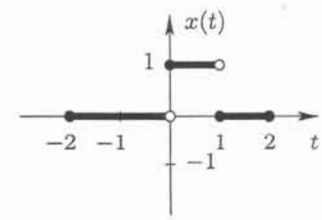
(a) Verify that ϕ_i for $i = 1, 2, 3$ are orthogonal with respect to the inner product

$$\langle f\,, g\rangle = \int_{-2}^{2} f(t)g(t)\,\mathrm{d}t.$$

(05pts)

(b) Find and plot the best representation of the function x, where

$$x(t) = \begin{cases} 0, & \text{if } -2 \le t < 0; \\ 1, & \text{if } 0 \le t < 1; \\ 0, & \text{if } 1 \le t \le 2. \end{cases}$$



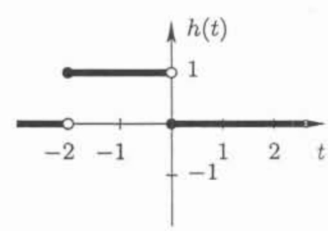
as a linear combination of ϕ_i for $i = 1, 2, 3$, such that the error based on the associated norm

$$\|f\| = \left[\int_{-2}^{2} (f(t))^2\,\mathrm{d}t\right]^{1/2},$$

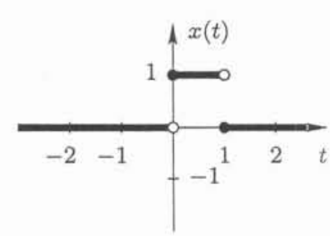
is minimized. (25pts)

4. For the following functions, h and x, determine and plot the convolution function $(h * x)$. (25pts)

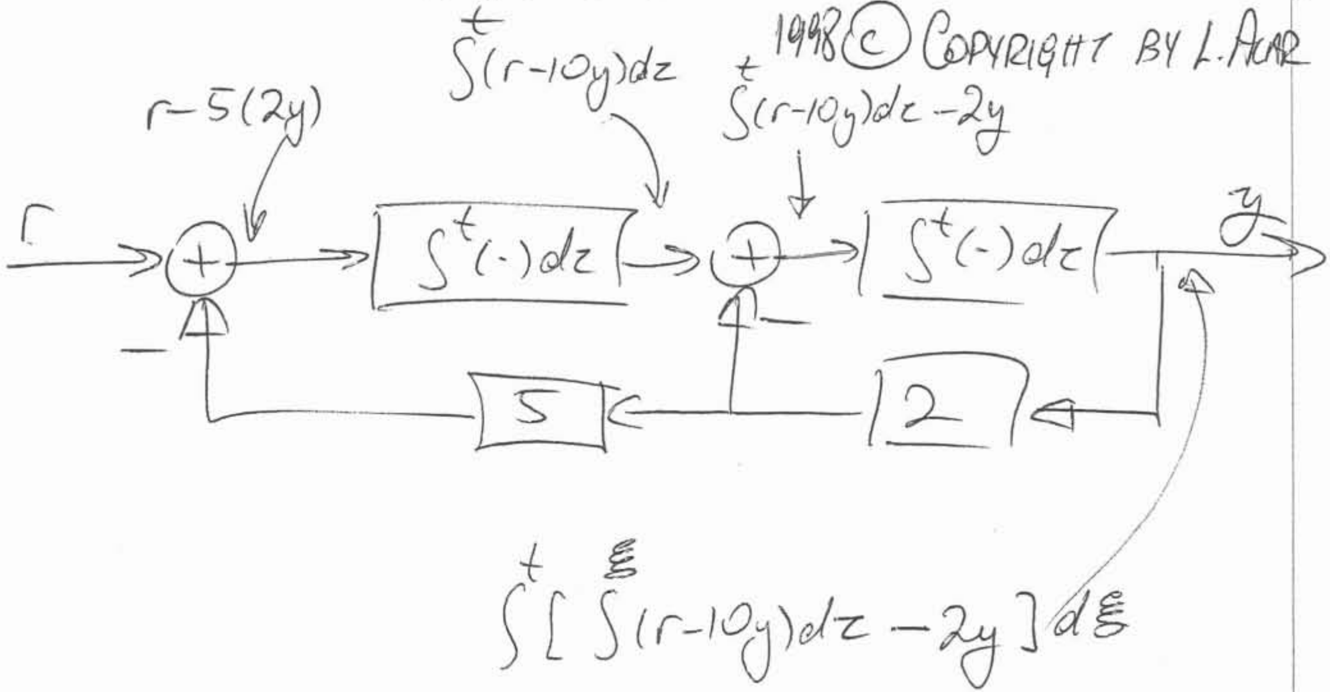
$$h(t) = \begin{cases} 1, & \text{if } -2 \le t < 0; \\ 0, & \text{elsewhere.} \end{cases}$$



$$x(t) = \begin{cases} 1, & \text{if } 0 \le t < 1; \\ 0, & \text{elsewhere.} \end{cases}$$



#1



$$y = \int^t \left[\int^\xi (r-10y)d\tau - 2y \right] d\xi$$

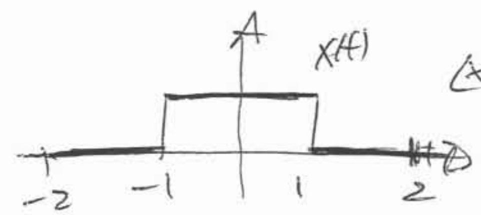
$$\dot{y} = \int^t (r-10y)d\tau - 2y$$

$$\ddot{y} = (r-10y) - 2\dot{y}$$

$$\ddot{y} + 2\dot{y} + 10y = r$$

#2

$$x(t) = \begin{cases} 0 & -2 \le t \le -1 \\ 1 & -1 < t < 1 \\ 0 & 1 \le t \le 2 \end{cases}$$



$T = 4$

$$\omega = \frac{2\pi}{T} = \frac{\pi}{2}$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t) \right]$$

$$a_n = \frac{\langle x(t), \cos(n\omega t) \rangle}{\langle \cos(n\omega t), \cos(n\omega t) \rangle} = \frac{2}{T} \int_T x(t) \cos(n\omega t)\, dt$$

$$b_n = \frac{\langle x(t), \sin(n\omega t) \rangle}{\langle \sin(n\omega t), \sin(n\omega t) \rangle} = \frac{2}{T} \int_T x(t) \sin(n\omega t)\, dt$$

$$a_n = \frac{2}{4} \int_{-2}^{2} x(t) \cos\left(\frac{\pi}{2} n t\right) dt$$

$$= \frac{1}{2} \int_{-1}^{1} \cos\left(\frac{\pi}{2} n t\right) dt$$

$$= \frac{1}{2} \left[\frac{1}{\pi/2\, n} \sin\left(\frac{\pi}{2} n t\right) \right]_{t=-1}^{1}, \quad n \ne 0$$

$$= \frac{2}{\pi n} \sin\left(\frac{\pi}{2} n\right), \quad n \ne 0$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} dt = 1$$

$$b_n = \frac{2}{4}\int_{-2}^{2} x(t)\sin\left(\frac{\pi}{2}nt\right)dt$$

$$= \frac{1}{2}\int_{-1}^{1}\sin\left(\frac{\pi}{2}nt\right)dt$$

$$= \frac{1}{2}\left\{-\frac{1}{\pi/2\, n}\cos\left(\frac{\pi}{2}nt\right)\right]_{t=-1}^{1},\quad n\neq 0$$

$$= 0$$

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n}\sin\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{2}nt\right)$$

Note: for $n = 4k$, $\frac{2}{\pi 4k}\sin\left(\frac{\pi}{2}4k\right) = \frac{1}{2\pi k}\sin(2\pi k) = 0$

$n = 4k+1$, $\frac{2}{\pi(4k+1)}\sin\left(\frac{\pi}{2}(4k+1)\right) = \frac{2}{\pi(4k+1)}\sin\left(2\pi k + \frac{\pi}{2}\right)$

$$= \frac{2}{\pi(4k+1)}$$

$n = 4k+2$, $\frac{2}{\pi(4k+2)}\sin\left(\frac{\pi}{2}(4k+2)\right) = 0$

$n = 4k+3$, $\frac{2}{\pi(4k+3)}\sin\left(\frac{\pi}{2}(4k+3)\right) = -\frac{2}{\pi(4k+3)}$

$$x(t) = 1 + \sum_{k=0}^{\infty} \left[\frac{2}{\pi(4k+1)} \cos\left(\frac{\pi}{2}(4k+1)t\right) - \frac{2}{\pi(4k+3)} \cos\left(\frac{\pi}{2}(4k+3)t\right) \right]$$

#3

$$\phi_1(t) = 1 \quad , -2 \le t \le 2$$

$$\phi_2(t) = \begin{cases} -1 , & -2 \le t < 0 \\ 1 , & 0 \le t \le 2 \end{cases}$$

$$\phi_3(t) = \begin{cases} -1 ; & -2 \le t < -1 \\ 1 ; & -1 \le t < 1 \\ -1 ; & 1 \le t \le 2 \end{cases}$$

a) $$\langle \phi_1, \phi_2 \rangle = \int_{-2}^{0} (1)(-1)\,dt + \int_{0}^{2} (1)(1)\,dt$$

$$= -2 + 2 = 0$$

$$\langle \phi_1, \phi_3 \rangle = \int_{-2}^{-1} (1)(-1)\,dt + \int_{-1}^{1} (1)(1)\,dt + \int_{1}^{2} (1)(-1)\,dt$$

$$= -1 + 2 + (-1) = 0$$

$$\langle \phi_2, \phi_3 \rangle = \int_{-2}^{-1} (-1)(-1)\,dt + \int_{-1}^{0} (-1)(1)\,dt + \int_{0}^{1} (1)(1)\,dt + \int_{1}^{2} (1)(-1)\,dt$$

$$= 1 - 1 + 1 - 1 = 0$$

So ϕ_1, ϕ_2, ϕ_3 are orthogonal b.ii

$$x(t) = \begin{cases} 0 & -2 \le t < 0 \\ 1 & 0 \le t < 1 \\ 0 & 1 \le t \le 2 \end{cases}$$

best approx. of $x(t)$ as a linear comb. of $\phi_i(t)$ is with the Fourier coefficients

$$x(t) = k_1 \phi_1(t) + k_2 \phi_2(t) + k_3 \phi_3(t)$$

$$k_1 = \frac{\langle x(t), \phi_1(t) \rangle}{\langle \phi_1(t), \phi_1(t) \rangle}$$

$$\langle x(t), \phi_1(t) \rangle = \int_0^1 (1)(1)\,dt = 1$$

$$\langle \phi_1(t), \phi_1(t) \rangle = \int_{-2}^{2} (1)(1)\,dt = 4$$

$$\Rightarrow k_1 = \frac{1}{4}$$

$$k_2 = \frac{\langle x(t), \phi_2(t) \rangle}{\langle \phi_2(t), \phi_2(t) \rangle}$$

$$\langle x(t), \phi_2(t) \rangle = \int_0^1 (1)(1)\,dt = 1$$

$$\langle \phi_2(t), \phi_2(t) \rangle = \int_{-2}^{0} (-1)(-1)\,dt + \int_0^2 (1)(1)\,dt \quad 2+2 = 4$$

$$\Rightarrow k_2 = \frac{1}{4}$$

$$k_3 = \frac{\langle x(t), \Phi_3(t)\rangle}{\langle \Phi_3(t), \Phi_3(t)\rangle}$$

$$\langle x(t), \Phi_3(t)\rangle = \int_0^1 (1)(1)\,dt = 1$$

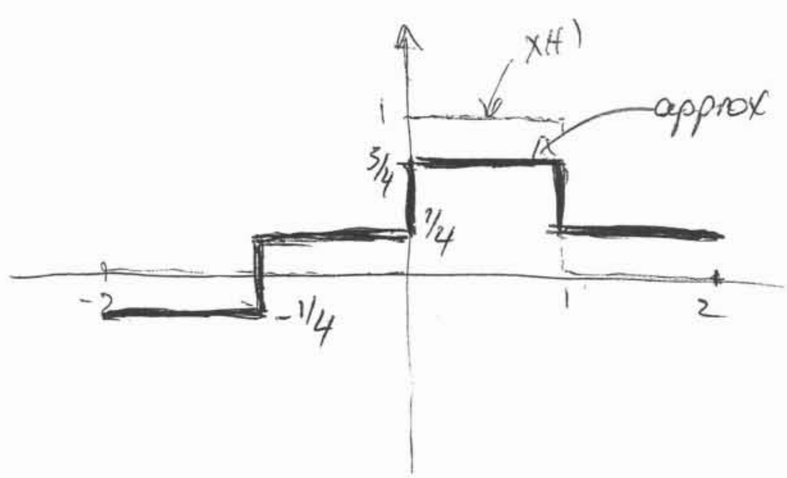
$$\langle \Phi_3(t), \Phi_3(t)\rangle = \int_{-2}^{-1} (-1)(-1)\,dt + \int_{-1}^{1} (1)(1)\,dt + \int_{1}^{2} (-1)(-1)\,dt$$

$$= 1 + 2 + 1 = 4$$

$$\Rightarrow k_3 = \frac{1}{4}$$

$$x(t) \approx \frac{1}{4}\left(\Phi_1(t) + \Phi_2(t) + \Phi_3(t)\right)$$

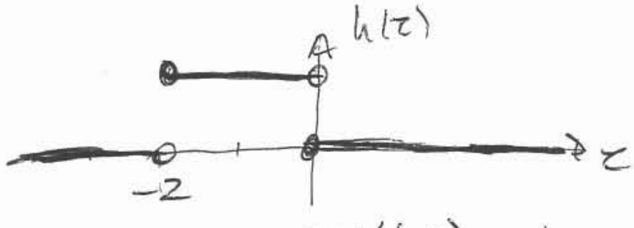
$$\approx \begin{cases} \frac{1}{4}(1 + (-1) + (-1)) = -\frac{1}{4}, & -2 \le t < -1 \\ \frac{1}{4}(1 + (-1) + 1) = \frac{1}{4}, & -1 \le t < 0 \\ \frac{1}{4}(1 + 1 + 1) = \frac{3}{4}, & 0 \le t < 1 \\ \frac{1}{4}(1 + 1 + (-1)) = \frac{1}{4}, & 1 \le t \le 2 \end{cases}$$

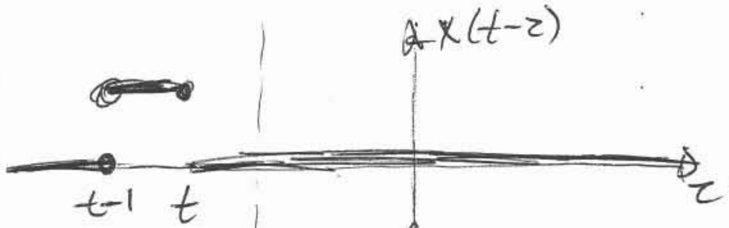


#4

$$h(t) = \begin{cases} 1, & -2 \leq t < 0 \\ 0, & \text{elsewhere} \end{cases}$$

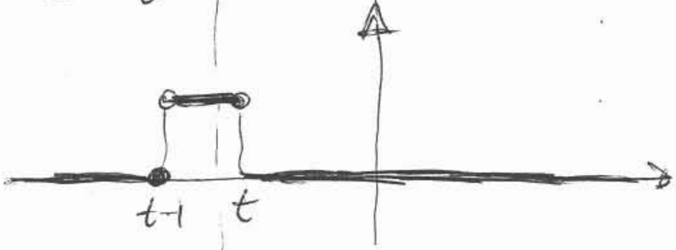
$$x(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & \text{elsewhere} \end{cases}$$





$(h*x)(t) = 0$

$t \leq -2$



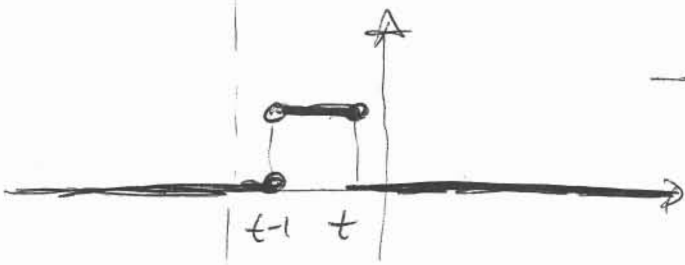
$$(h*x)(t) = \int_{-2}^{t} (1)(1)\,d\tau = t+2$$

$t-1 \leq -2 \leq t$

$-1 \leq -2-t \leq 0$

$1 \leq -t \leq 2$

$-2 \leq t \leq -1$



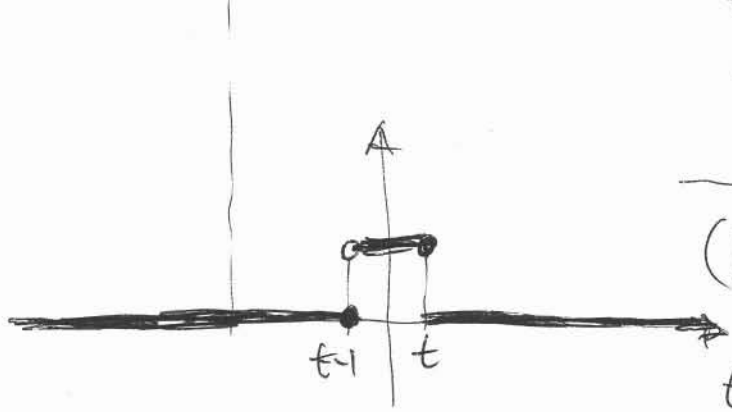
$$(h*x)(t) = \int_{t-1}^{t} (1)(1)\,d\tau$$

$$= t-(t-1) = 1$$

$t-1 \geq -2, \quad t \geq -1$

$t \leq 0$

$-1 \leq t \leq 0$

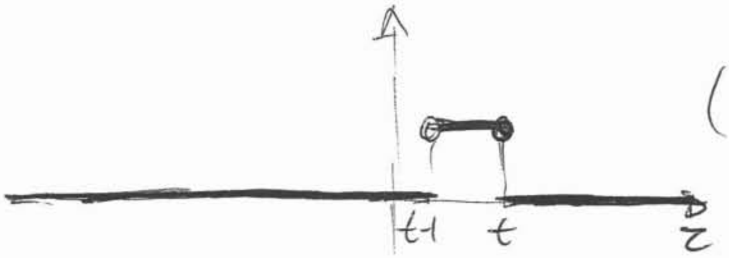


$$(h*x)(t) = \int_{t-1}^{0} (1)(1)\,d\tau = -t+1$$

$t-1 \leq 0 \leq t$

$-1 \leq -t \leq 0$

$0 \leq t \leq 1$



$(h*x)(t) = 0$

$0 \leq t-1$

$1 \leq t$

$$(h*x)(t) = \begin{cases} 0, & t \leq -2 \\ t+2, & -2 \leq t \leq -1 \\ 1, & -1 \leq t \leq 0 \\ -t+1, & 0 \leq t \leq 1 \\ 0, & 1 \leq t \end{cases}$$