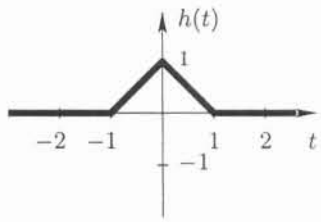
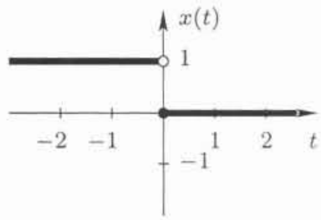
EE 265 | Exam#2 50 minutes | Mar. 21, 2007

1. For the following functions, h and x, determine and plot the convolution function $\big(h * x\big)$. (30pts)

$$h(t) = \begin{cases} t+1, & \text{if } -1 \le t < 0; \\ -t+1, & \text{if } 0 \le t \le 1; \\ 0, & \text{elsewhere.} \end{cases}$$

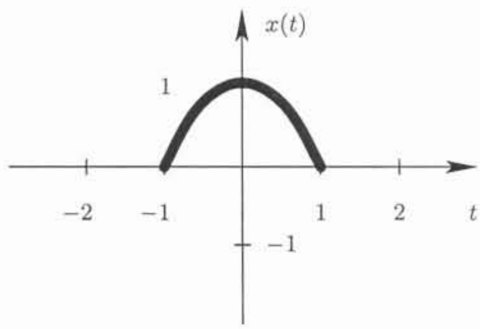


$$x(t) = \begin{cases} 1, & \text{if } t < 0; \\ 0, & \text{if } t \ge 0. \end{cases}$$



2. Consider the following function $x(t)$, where $-1 \le t \le 1$.

$x(t) = \cos\big((\pi/2)t\big)$ for $-1 \le t \le 1$.



(a) Determine the trigonometric fourier-series of $x(t)$ for $-1 \le t \le 1$. (10pts)

(b) Determine and sketch its single-sided frequency representation. (10pts)

(c) Determine the complex-exponential fourier-series of $x(t)$ for $-1 \le t \le 1$. (20pts)

(d) Determine and sketch its double-sided frequency representation. (10pts)

HINT: The following indefinite integrals may be useful.

$$\int^t \cos(\omega\tau)\,\mathrm{d}\tau = \frac{\sin(\omega t)}{\omega}. \qquad \int^t \sin(\omega\tau)\,\mathrm{d}\tau = -\frac{\cos(\omega t)}{\omega}.$$

$$\int^t \cos^2(\omega\tau)\,\mathrm{d}\tau = \frac{t}{2} + \frac{\sin(2\omega t)}{4\omega}. \qquad \int^t \sin^2(\omega\tau)\,\mathrm{d}\tau = \frac{t}{2} - \frac{\sin(2\omega t)}{4\omega}.$$

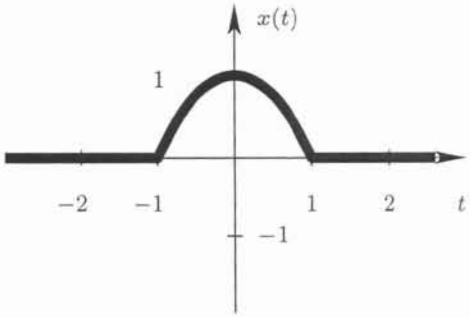
$$\int^t \cos(\omega_1\tau)\cos(\omega_2\tau)\,\mathrm{d}\tau = \frac{\sin\big((\omega_1-\omega_2)t\big)}{2(\omega_1-\omega_2)} + \frac{\sin\big((\omega_1+\omega_2)t\big)}{2(\omega_1+\omega_2)} \text{ for } |\omega_1| \neq |\omega_2|.$$

$$\int^t \cos(\omega_1\tau)\sin(\omega_2\tau)\,\mathrm{d}\tau = -\frac{\cos\big((\omega_1-\omega_2)t\big)}{2(\omega_1-\omega_2)} - \frac{\cos\big((\omega_1+\omega_2)t\big)}{2(\omega_1+\omega_2)} \text{ for } |\omega_1| \neq |\omega_2|.$$

$$\int^t e^{\alpha\tau}\cos(\omega\tau)\,\mathrm{d}\tau = \frac{e^{\alpha t}\big(\alpha\cos(\omega t) + \omega\sin(\omega t)\big)}{\alpha^2+\omega^2}. \qquad \int^t e^{\alpha\tau}\sin(\omega\tau)\,\mathrm{d}\tau = \frac{e^{\alpha t}\big(\alpha\sin(\omega t) - \omega\cos(\omega t)\big)}{\alpha^2+\omega^2}.$$

3. Determine the fourier transform of $x(t)$ for $-\infty \leq t \leq \infty$, where

$$x(t) = \begin{cases} \cos\big((\pi/2)t\big), & \text{if } -1 \leq t \leq 1; \\ 0, & \text{elsewhere.} \end{cases}$$



(20pts)

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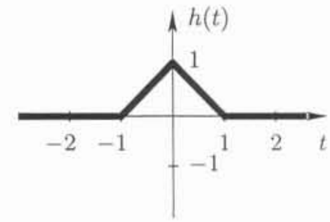
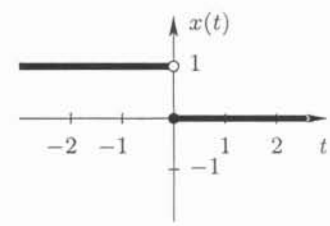
Solutions

1. For the following functions, h and x, determine and plot the convolution function $\big(h * x\big)$.

$$h(t) = \begin{cases} t+1, & \text{if } -1 \le t < 0; \\ -t+1, & \text{if } 0 \le t \le 1; \\ 0, & \text{elsewhere.} \end{cases}$$



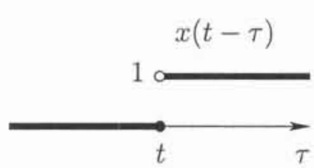
$$x(t) = \begin{cases} 1, & \text{if } t < 0; \\ 0, & \text{if } t \ge 0. \end{cases}$$

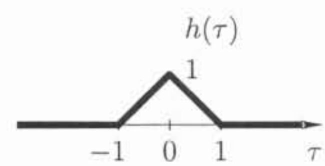


Solution: We may use either the graphical method or the analytical method to determine the convolution. However, in this case, the graphical method is simpler.

$$\big(h * x\big)(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau.$$

Using the second expression in the definition of the convolution, we first obtain $x(t-\tau)$ along with $h(\tau)$.

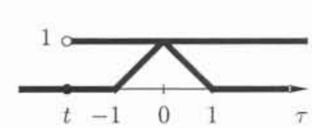




Then, integrate the product of $x(t-\tau)$ and $h(\tau)$ for different values of t.

We get

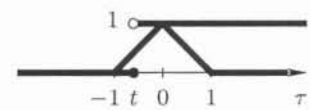
$$\begin{aligned}\big(h * x\big)(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau \\ &= \int_{-1}^{1} h(\tau)(1)\,\mathrm{d}\tau = 1,\end{aligned}$$



from the area of the triangle, when $t \le -1$.

Next,

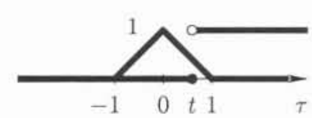
$$
\begin{aligned}
(h*x)(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau = \int_{t}^{1} h(\tau)(1)\,\mathrm{d}\tau \\
&= \int_{t}^{0} (\tau+1)(1)\,\mathrm{d}\tau + \int_{0}^{1} (-\tau+1)(1)\,\mathrm{d}\tau \\
&= \left[\frac{\tau^2}{2}+\tau\right]_{\tau=t}^{\tau=0} + \frac{1}{2} \\
&= \left[\left(\frac{(0)^2}{2}+(0)\right) - \left(\frac{(t)^2}{2}+(t)\right)\right] + \frac{1}{2} \\
&= -\frac{t^2}{2} - t + \frac{1}{2},
\end{aligned}
$$



when $-1 < t \leq 0$.

Then,

$$
\begin{aligned}
(h*x)(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau = \int_{t}^{1} h(\tau)(1)\,\mathrm{d}\tau \\
&= \int_{t}^{1} (-\tau+1)(1)\,\mathrm{d}\tau = \left[-\frac{\tau^2}{2}+\tau\right]_{\tau=t}^{\tau=1} \\
&= \left[\left(-\frac{(1)^2}{2}+(1)\right) - \left(-\frac{(t)^2}{2}+(t)\right)\right] \\
&= \frac{t^2}{2} - t + \frac{1}{2},
\end{aligned}
$$



when $0 < t \leq 1$.

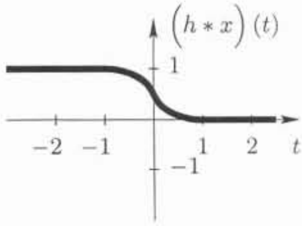
Finally,

$$
(h*x)(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau = 0,
$$

when $1 < t$.

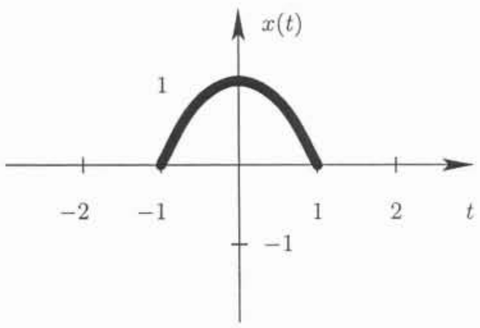
Therefore,

$$\big(h*x\big)(t) = \begin{cases} 1, & \text{if } t \le -1; \\ -t^2/2 - t + 1/2, & \text{if } -1 < t \le 0; \\ t^2/2 - t + 1/2, & \text{if } 0 < t \le 1; \\ 0, & \text{if } 1 < t. \end{cases}$$



2. Consider the following function $x(t)$, where $-1 \le t \le 1$.

$x(t) = \cos\big((\pi/2)t\big)$ for $-1 \le t \le 1$.



(a) Determine the trigonometric fourier-series of $x(t)$ for $-1 \le t \le 1$.

Solution: The trigonometric fourier-series of x is in the form of an infinite sum, such that

$$x(t) = a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(n\omega t) + b_n \sin(n\omega t)\big),$$

where

$$a_0 = \frac{\langle x\,,1\rangle}{\langle 1\,,1\rangle} = \frac{1}{T}\int_T x(t)\,\mathrm{d}t,$$

$$a_n = \frac{\langle x\,,\,\cos(n\omega(\cdot))\rangle}{\langle \cos(n\omega(\cdot))\,,\,\cos(n\omega(\cdot))\rangle} = \frac{2}{T}\int_T x(t)\cos(n\omega t)\,\mathrm{d}t,$$

$$b_n = \frac{\langle x\,,\,\sin(n\omega(\cdot))\rangle}{\langle \sin(n\omega(\cdot))\,,\,\sin(n\omega(\cdot))\rangle} = \frac{2}{T}\int_T x(t)\sin(n\omega t)\,\mathrm{d}t$$

for $n \ge 1$, and

$$\langle f\,,g\rangle = \int_T f(t)g(t)\,\mathrm{d}t.$$

In our case $T = 2$ and $\omega = 2\pi/T = \pi$, so

$$a_0 = \frac{1}{2}\int_{-1}^{1} x(t)\,\mathrm{d}t = \frac{1}{2}\int_{-1}^{1} \cos\big((\pi/2)t\big)\,\mathrm{d}t = \frac{1}{2}\left[\frac{\sin\big((\pi/2)t\big)}{\pi/2}\right]_{t=-1}^{t=1}$$
$$= \frac{1}{2}\left[\left(\frac{\sin\big((\pi/2)(1)\big)}{\pi/2}\right) - \left(\frac{\sin\big((\pi/2)(-1)\big)}{\pi/2}\right)\right] = \frac{2}{\pi},$$

$$a_n = \frac{2}{2}\int_{-1}^{1} x(t)\cos(n\omega t)\,\mathrm{d}t = \int_{-1}^{1} \cos\big((\pi/2)t\big)\cos\big(n\pi t\big)\,\mathrm{d}t$$
$$= \left[\frac{\sin\big((\pi/2 - n\pi)t\big)}{2(\pi/2 - n\pi)} + \frac{\sin\big((\pi/2 + n\pi)t\big)}{2(\pi/2 + n\pi)}\right]_{t=-1}^{t=1}$$
$$= \Bigg[\left(\frac{\sin\big((\pi/2 - n\pi)(1)\big)}{2(\pi/2 - n\pi)} + \frac{\sin\big((\pi/2 + n\pi)(1)\big)}{2(\pi/2 + n\pi)}\right) - \left(\frac{\sin\big((\pi/2 - n\pi)(-1)\big)}{2(\pi/2 - n\pi)} + \frac{\sin\big((\pi/2 + n\pi)(-1)\big)}{2(\pi/2 + n\pi)}\right)\Bigg]$$
$$= \frac{2\sin\big((\pi/2 - n\pi)\big)}{2(\pi/2 - n\pi)} + \frac{2\sin\big((\pi/2 + n\pi)\big)}{2(\pi/2 + n\pi)} = \frac{(-1)^n}{(1-2n)(\pi/2)} + \frac{(-1)^n}{(1+2n)(\pi/2)}$$
$$= (-1)^n\frac{2}{\pi}\left(\frac{1}{1-2n} + \frac{1}{1+2n}\right) = (-1)^n\frac{2}{\pi}\left(\frac{2}{1-4n^2}\right) = (-1)^n\left(\frac{4/\pi}{1-4n^2}\right),$$

and

$$b_n = \frac{2}{2}\int_{-1}^{1} x(t)\sin(n\omega t)\,\mathrm{d}t = \int_{-1}^{1} \cos\big((\pi/2)t\big)\sin\big(n\pi t\big)\,\mathrm{d}t$$
$$= \left[-\frac{\cos\big((\pi/2 - n\pi)t\big)}{2(\pi/2 - n\pi)} - \frac{\cos\big((\pi/2 + n\pi)t\big)}{2(\pi/2 + n\pi)}\right]_{t=-1}^{t=1}$$
$$= \Bigg[\left(-\frac{\cos\big((\pi/2 - n\pi)(1)\big)}{2(\pi/2 - n\pi)} - \frac{\cos\big((\pi/2 + n\pi)(1)\big)}{2(\pi/2 + n\pi)}\right) - \left(-\frac{\cos\big((\pi/2 - n\pi)(-1)\big)}{2(\pi/2 - n\pi)} - \frac{\cos\big((\pi/2 + n\pi)(-1)\big)}{2(\pi/2 + n\pi)}\right)\Bigg] = 0.$$

So,

$$x(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n 4}{(1-4n^2)\pi}\cos(n\pi t) \text{ for } -1 \le t \le 1.$$

(b) Determine and sketch its single-sided frequency representation.

Solution: Since the trigonometric fourier-series of $x(t)$ is

$$x(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n 4}{(1-4n^2)\pi}\cos(n\pi t),$$

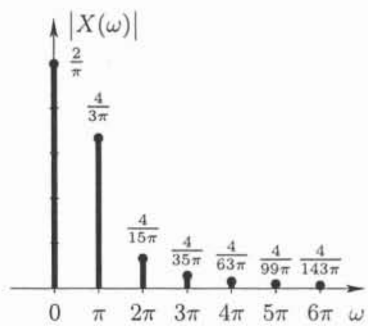
we can determine its single-sided frequency representation from the magnitudes and phases of the cosine-term coefficients. Here, since all the sine terms have zero coefficients, we don't have the extra step of combining the cosine and sine terms with the same frequencies into single cosine terms with phase angles. However, negative cosine-coefficients give an additional phase angle of π, since

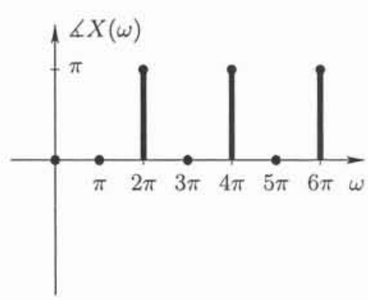
$$-\cos(\omega t) = \cos(\omega t + \pi).$$

In our case, the coefficient is negative for an even positive integer n. Therefore,

$$\left|X(\omega)\right|_{\omega=n\pi} = \begin{cases} 2/\pi, & \text{if } n = 0; \\ 4/(|1-4n^2|\pi), & \text{if } n \text{ is a positive integer.} \end{cases}$$

$$\measuredangle X(\omega)\Big|_{\omega=n\pi} = \begin{cases} 0, & \text{if } n = 0 \text{ or } n \text{ is an odd positive integer;} \\ \pi, & \text{if } n \text{ is an even positive integer.} \end{cases}$$





(c) Determine the complex-exponential fourier-series of $x(t)$ for $-1 \leq t \leq 1$.

Solution: The complex-exponential fourier-series of $x(t)$ is given by

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t},$$

where $\omega = 2\pi/T$ and

$$c_n = \frac{1}{T}\int_T x(t)e^{-jn\omega t}\,\mathrm{d}t.$$

In our case; $T = 2$, $\omega = 2\pi/2 = \pi$, and

$$\begin{aligned}
c_n &= \frac{1}{T}\int_T x(t)e^{-jn\omega t}\,\mathrm{d}t \\
&= \frac{1}{2}\int_{-1}^{1}\cos\big((\pi/2)t\big)e^{-jn\omega t}\,\mathrm{d}t \\
&= \frac{1}{2}\left[\frac{e^{-jn\omega t}\Big(-jn\omega\cos\big((\pi/2)t\big)+(\pi/2)\sin\big((\pi/2)t\big)\Big)}{(-jn\omega)^2+(\pi/2)^2}\right]_{t=-1}^{t=1} \\
&= \frac{1}{2}\left[\left(\frac{e^{-jn\omega(1)}\Big(-jn\omega\cos\big((\pi/2)(1)\big)+(\pi/2)\sin\big((\pi/2)(1)\big)\Big)}{(-jn\omega)^2+(\pi/2)^2}\right)\right. \\
&\qquad\left. -\left(\frac{e^{-jn\omega(-1)}\Big(-jn\omega\cos\big((\pi/2)(-1)\big)+(\pi/2)\sin\big((\pi/2)(-1)\big)\Big)}{(-jn\omega)^2+(\pi/2)^2}\right)\right] \\
&= \frac{1}{2}\left[\left(\frac{e^{-jn\pi}\big(\pi/2\big)}{-n^2\pi^2+\pi^2/4}\right)-\left(\frac{-e^{jn\pi}\big(\pi/2\big)}{-n^2\pi^2+\pi^2/4}\right)\right] \\
&= \left(\frac{\pi}{4}\right)\left(\frac{1}{\pi^2/4-n^2\pi^2}\right)\left(e^{-jn\pi}+e^{jn\pi}\right) = \frac{2/\pi}{1-4n^2}\left(\frac{e^{jn\pi}+e^{-jn\pi}}{2}\right) = \frac{2/\pi}{1-4n^2}\cos(n\pi).
\end{aligned}$$

Since $\cos(n\pi) = (-1)^n$, the complex-exponential fourier-series of $x(t)$ is

$$x(t) = \sum_{n=-\infty}^{\infty}\frac{(-1)^n 2}{(1-4n^2)\pi}e^{jn\pi t}\text{ for } -1 \le t \le 1.$$

(d) Determine and sketch its double-sided frequency representation.

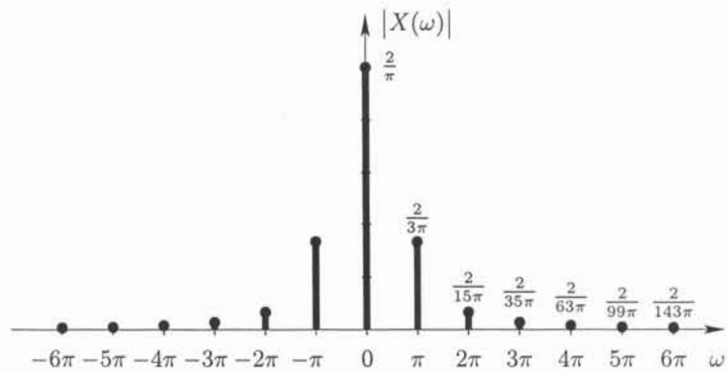
Solution: Since the complex-exponential fourier-series of $x(t)$ is

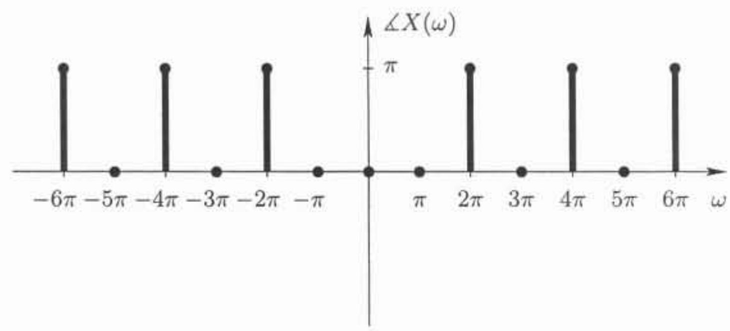
$$x(t) = \sum_{n=-\infty}^{\infty}\frac{(-1)^n 2}{(1-4n^2)\pi}e^{jn\pi t},$$

we can determine its double-sided frequency representation from the magnitudes and phases of the complex coefficients. Since the term $(1-4n^2)$ is negative except for $n = 0$, we get

$$\big|X(\omega)\big|_{\omega=n\pi} = \frac{2}{|1-4n^2|\pi}\text{ for all integer } n\text{, and}$$

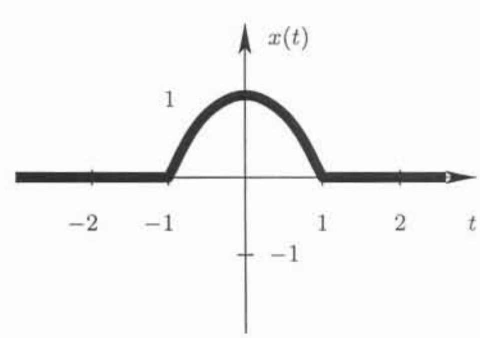
$$\measuredangle X(\omega)\Big|_{\omega=n\pi} = \begin{cases}0, & \text{if } n=0 \text{ or } n \text{ is an odd integer;}\\ \pi, & \text{if } n\neq 0 \text{ and } n \text{ is an even integer.}\end{cases}$$





3. Determine the fourier transform of $x(t)$ for $-\infty \le t \le \infty$, where

$$x(t) = \begin{cases} \cos\big((\pi/2)t\big), & \text{if } -1 \le t \le 1; \\ 0, & \text{elsewhere.} \end{cases}$$



Solution: The fourier transform of x is

$$\mathcal{F}\left[\, x \,\right](\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\,\mathrm{d}t = \int_{-1}^{1} \cos\big((\pi/2)t\big) e^{-j\omega t}\,\mathrm{d}t$$

$$= \left[\frac{e^{-j\omega t}\Big(-j\omega\cos\big((\pi/2)t\big) + (\pi/2)\sin\big((\pi/2)t\big)\Big)}{(-j\omega)^2 + (\pi/2)^2} \right]_{t=-1}^{t=1}$$

$$= \left[\frac{e^{-j\omega(1)}\Big(-j\omega\cos\big((\pi/2)(1)\big) + (\pi/2)\sin\big((\pi/2)(1)\big)\Big)}{-\omega^2 + \pi^2/4} \right] - \left[\frac{e^{-j\omega(-1)}\Big(-j\omega\cos\big((\pi/2)(-1)\big) + (\pi/2)\sin\big((\pi/2)(-1)\big)\Big)}{-\omega^2 + \pi^2/4} \right]$$

$$= \left[\frac{e^{-j\omega}\big(\pi/2\big)}{-\omega^2 + \pi^2/4} \right] - \left[\frac{e^{j\omega}\big(-\pi/2\big)}{-\omega^2 + \pi^2/4} \right] = \frac{4\pi}{\pi^2 - 4\omega^2}\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right),$$

or

$$\mathcal{F}\left[\, x \,\right](\omega) = \frac{4\pi}{\pi^2 - 4\omega^2}\cos(\omega).$$