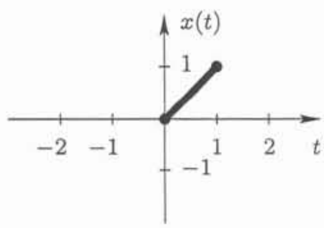
EE 261 — **Exam#2** — **Apr. 15, 1998**

50 minutes

1. Determine the complex-exponential fourier series of $x(t)$ for $0 \le t \le 1$, where

$$x(t) = t \text{ for } 0 \le t \le 1.$$



HINT: $\int^t \tau e^{\alpha\tau}\, d\tau = (\alpha t - 1)e^{\alpha t}/\alpha^2$. (25pts)

2. Determine the fourier transform of the function x, where

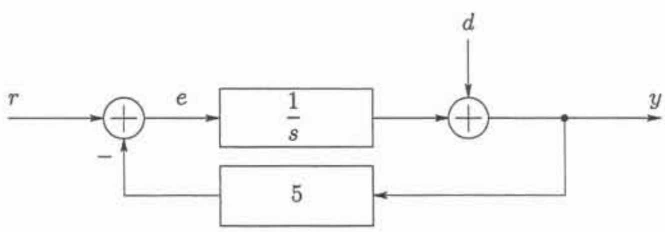
$$x(t) = e^{-a|t|} \text{ for } -\infty < t < \infty$$

and $a > 0$. (25pts)

3. In the following block diagram, the laplace transform of the output, under zero initial-condition, is

$$\mathcal{L}\left[\, y \,\right](s) = Y(s) = \left(\frac{2}{s+5}\right)\left(\frac{6s}{s^2+9} + \frac{1}{s+1}\right),$$

when the reference input is $r(t) = 2e^{-t}\mathbb{1}(t)$. Determine the signal $d(t)$ for $t \ge 0$. HINT: First obtain $Y(s)$ in terms of $\mathcal{L}\left[\, d \,\right](s) = D(s)$ and $\mathcal{L}\left[\, r \,\right](s) = R(s)$, then solve for $D(s)$. (25pts)



4. Find the inverse laplace transform of the following function.

$$F(s) = \frac{2s^2 + 5s + 39}{(s+1)(s^2+2s+10)}.$$

(25pts)

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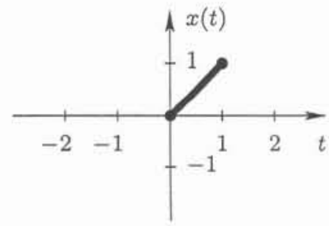
Solutions

1. Determine the complex-exponential fourier series of $x(t)$ for $0 \le t \le 1$, where

$$x(t) = t \text{ for } 0 \le t \le 1.$$



HINT: $\int^t \tau e^{\alpha\tau}\, \mathrm{d}\tau = (\alpha t - 1)e^{\alpha t}/\alpha^2$.

Solution: The complex-exponential fourier series of x is in the form of an infinite sum, such that

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t},$$

where

$$c_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega t}\, \mathrm{d}t.$$

In our case $T = 1$ and $\omega = 2\pi/T = 2\pi$, so

$$c_n = \int_0^1 x(t)e^{-j2\pi nt}\, \mathrm{d}t = \int_0^1 te^{-j2\pi nt}\, \mathrm{d}t = \left(\frac{-j2\pi nt - 1}{(-j2\pi n)^2}e^{-j2\pi nt}\right)_{t=0}^{t=1}$$
$$= \left(\frac{-j2\pi n - 1}{-(2\pi n)^2}e^{-j2\pi n}\right) - \left(\frac{-1}{-(2\pi n)^2}\right) \text{ for } n \neq 0.$$

Since $e^{-j2\pi n} = 1$ for all integer n,

$$c_n = \frac{1}{(2\pi n)^2}(j2\pi n + 1 - 1) = j\left(\frac{1}{2\pi n}\right) \text{ for } n \neq 0..$$

For $n = 0$, we get

$$c_0 = \int_0^1 x(t)\, \mathrm{d}t = \int_0^1 t\, \mathrm{d}t = \left(\frac{t^2}{2}\right)_{t=0}^{t=1} = \frac{1}{2}.$$

Therefore,

$$x(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} j\left(\frac{1}{2\pi n}\right)e^{j2\pi nt} \text{ for } 0 \le t \le 1.$$

2. Determine the fourier transform of the function x, where

$$x(t) = e^{-a|t|} \text{ for } -\infty < t < \infty$$

and $a > 0$.

Solution: The fourier transform of x is

$$\begin{aligned}\mathcal{F}\left[x\right](\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,\mathrm{d}t = \int_{-\infty}^{\infty} e^{-a|t|}e^{-j\omega t}\,\mathrm{d}t = \int_{-\infty}^{0} e^{at}e^{-j\omega t}\,\mathrm{d}t + \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,\mathrm{d}t \\ &= \left(\frac{e^{(a-j\omega)t}}{(a-j\omega)}\right)_{t=-\infty}^{t=0} + \left(\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right)_{t=0}^{t=\infty} = \left(\frac{1}{(a-j\omega)} - 0\right) + \left(0 - \frac{1}{-(a+j\omega)}\right) \\ &= \frac{1}{(a-j\omega)} + \frac{1}{(a+j\omega)},\end{aligned}$$

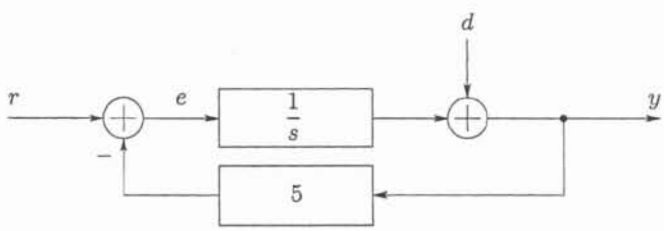
or

$$\mathcal{F}\left[x\right](\omega) = \frac{2a}{a^2+\omega^2}.$$

3. In the following block diagram, the laplace transform of the output, under zero initial-condition, is

$$\mathcal{L}\left[y\right](s) = Y(s) = \left(\frac{2}{s+5}\right)\left(\frac{6s}{s^2+9} + \frac{1}{s+1}\right),$$

when the reference input is $r(t) = 2e^{-t}\mathbb{1}(t)$. Determine the signal $d(t)$ for $t \geq 0$. HINT: First obtain $Y(s)$ in terms of $\mathcal{L}\left[d\right](s) = D(s)$ and $\mathcal{L}\left[r\right](s) = R(s)$, then solve for $D(s)$.



Solution: From the block diagram, we get

$$Y(s) = D(s) + V(s) = D(s) + \left(\frac{1}{s}\right)E(s) = D(s) + \left(\frac{1}{s}\right)(R(s) - 5Y(s)),$$

or

$$D(s) = Y(s) + \left(\frac{5}{s}\right)Y(s) - \left(\frac{1}{s}\right)R(s) = \left(\frac{s+5}{s}\right)Y(s) - \left(\frac{1}{s}\right)R(s).$$

Since,

$$Y(s) = \left(\frac{2}{s+5}\right)\left(\frac{6s}{s^2+9} + \frac{1}{s+1}\right),$$

and

$$r(t) = 2e^{-t}\mathbb{1}(t) \longleftrightarrow 2\frac{1}{s+1} = R(s),$$

we have

$$\begin{aligned} D(s) &= \left(\frac{s+5}{s}\right) Y(s) - \left(\frac{1}{s}\right) R(s) \\ &= \left(\frac{s+5}{s}\right)\left(\frac{2}{s+5}\right)\left(\frac{6s}{s^2+9} + \frac{1}{s+1}\right) - \left(\frac{1}{s}\right)\left(\frac{2}{s+1}\right) \\ &= \frac{12}{s^2+9} + \frac{2}{s(s+1)} - \frac{2}{s(s+1)} = \frac{12}{s^2+9} = 4\frac{3}{s^2+3^2}. \end{aligned}$$

Therefore,

$$d(t) = 4\sin(3t)\mathbb{1}(t).$$

4. Find the inverse laplace transform of the following function.

$$F(s) = \frac{2s^2+5s+39}{(s+1)(s^2+2s+10)}.$$

Solution: We first write the partial fraction expansion of $F(s)$.

$$F(s) = \frac{2s^2+5s+39}{(s+1)(s^2+2s+10)} = \frac{\alpha}{s+1} + \frac{\beta_1 s + \beta_2}{s^2+2s+10}.$$

Here,

$$\alpha = \lim_{s \to -1} [(s+1)F(s)] = \left[\frac{2s^2+5s+39}{s^2+2s+10}\right]_{s=-1} = \frac{36}{9} = 4.$$

Then,

$$F(s) = \frac{4}{s+1} + \frac{\beta_1 s + \beta_2}{s^2+2s+10}.$$

To find β_1 and β_2, first we let $s = 0$ in the above equation, i.e.,

$$\frac{39}{(1)(10)} = \frac{4}{1} + \frac{\beta_2}{10}.$$

As a result, $\beta_2 = -1$, and

$$F(s) = \frac{4}{s+1} + \frac{\beta_1 s - 1}{s^2+2s+10}.$$

Next, we let $s = 1$, i.e.,

$$\frac{2(1)^2+5(1)+39}{(1+1)((1)^2+2(1)+10)} = \frac{4}{(1)+1} + \frac{\beta_1(1)-1}{(1)^2+2(1)+10},$$

and we obtain $\beta_1 = -2$. So,

$$F(s) = \frac{4}{s+1} + \frac{-2s-1}{s^2+2s+10}.$$

We know that

$$e^{-\alpha t}\mathbb{1}(t) \longleftrightarrow \frac{1}{s+\alpha},$$

$$\sin(\omega t)\mathbb{1}(t) \longleftrightarrow \frac{\omega}{s^2+\omega^2},$$

$$\cos(\omega t)\mathbb{1}(t) \longleftrightarrow \frac{s}{s^2+\omega^2},$$

and

$$e^{-\alpha t}f(t) \longleftrightarrow F(s+\alpha).$$

Therefore, we complete the second-order denominator to squares, and arrange the terms to fit into the sinusoidal function transforms.

$$\begin{aligned} F(s) &= \frac{4}{s+1} + \frac{-2s-1}{s^2+2s+10} = \frac{4}{s+1} + \frac{-2s-1}{(s^2+2s+1)-1+10} = \frac{4}{s+1} + \frac{-2s-1}{(s+1)^2+9} \\ &= \frac{4}{s+1} + \frac{-2(s+1)-(-2)(1)-1}{(s+1)^2+3^2} = \frac{4}{s+1} + \frac{-2(s+1)+1}{(s+1)^2+3^2} \\ &= (4)\frac{1}{s+1} - (2)\frac{(s+1)}{(s+1)^2+3^2} + (1/3)\frac{3}{(s+1)^2+3^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{L}^{-1}\left[F\right](t) &= \left(4e^{-t} - 2e^{-t}\cos(3t) + (1/3)e^{-t}\sin(3t)\right)\mathbb{1}(t), \\ &= 4e^{-t} - 2e^{-t}\cos(3t) + (1/3)e^{-t}\sin(3t) \text{ for } t \geq 0. \end{aligned}$$