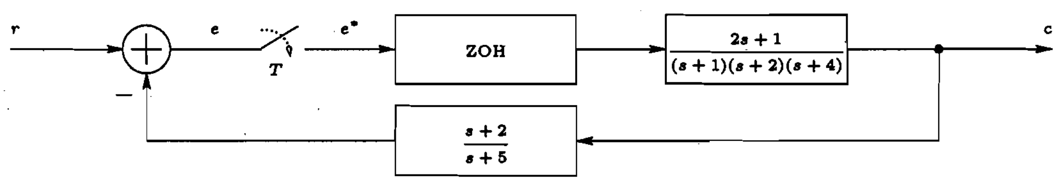
**EE 331** **Exam#1** **Mar. 20, 2008**

75 minutes



1. For the following system, determine the transfer function, assuming $T = 1$ s. (20pts)

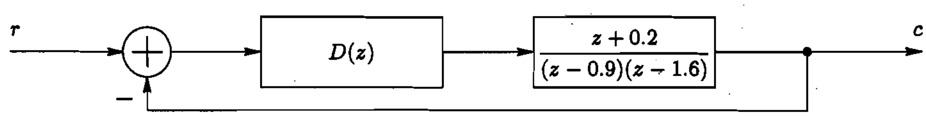



2. Consider a negative feedback discrete-time control system, where the loop gain is given by

$$G(z)H(z) = K\frac{z + 0.5}{10z^3 + 4z^2 - 8.4z + 1.44}.$$

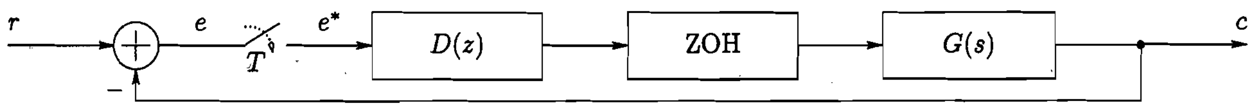
Determine the range of stability in terms of the gain $K$. (25pts)

3. Consider the following system with a sampling period of 1/2 second.

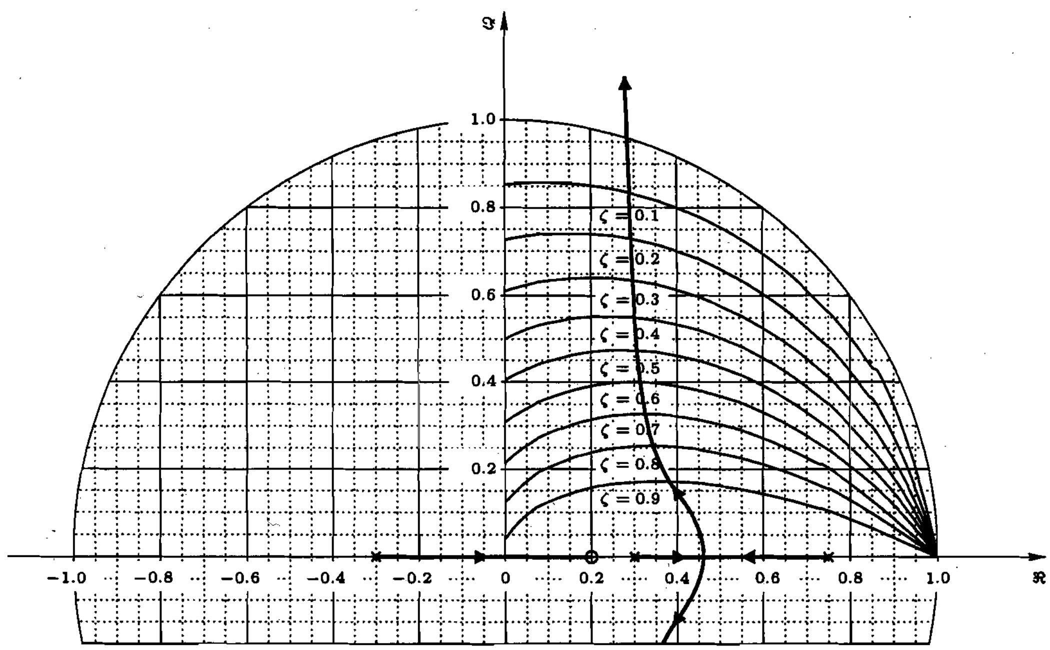



Design the simplest controller $D(z)$, such that the 2% settling-time is about 3 seconds, and the maximum percent-overshoot is approximately 10% for the unit-step input. (30pts)

4. Consider the following control system. when $D(z) = K$.




The root-locus diagram for the discrete-time representation of the system is given below, when $D(z) = K$.




$$z = \Re(z) + j\Im(z) = e^{-\zeta\omega T}e^{j\sqrt{1-\zeta^2}\omega T}$$

(a) Design the simplest controller, such that the dominant closed-loop poles have $\zeta = 0.4$. (15pts)

(b) Specify the resulting $C(z)/R(z)$. All the unknown quantities must be evaluated, and each pole must be specified individually. (10pts)

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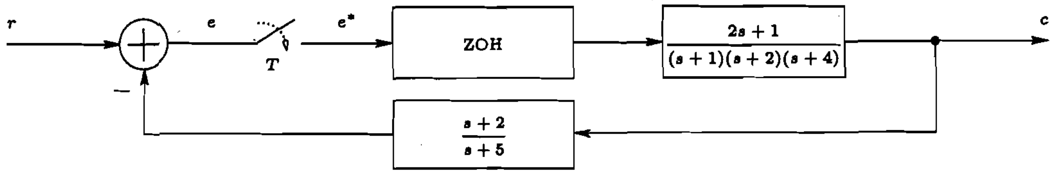
1. For the following system, determine the transfer function, assuming $T = 1\,\mathrm{s}$.




**Solution:** In order to be able to take the z-transforms of signals, they need to be sampled or pseudo-sampled. Denoting the transfer function of the zero-order hold (ZOH) by $G_{\mathrm{ZOH}}$, we have

$$E(s) = R(s) - \left(\frac{s+2}{s+5}\right)\left(\frac{2s+1}{(s+1)(s+2)(s+4)}\right) G^*_{\mathrm{ZOH}}(s)E^*(s),$$

where $E^*(s)$ represents the ideally-sampled $E(s)$. When we take the z-transforms of the inverse laplace transforms in the above equation, we get

$$E(z) = R(z) - \mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\left(\tfrac{2s+1}{(s+1)(s+4)(s+5)}\right) G^*_{\mathrm{ZOH}}(s)\right]\right](z)E(z).$$

To simplify the notation, we let

$$\begin{aligned}(HGG_{\mathrm{ZOH}})(z) &= \mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\left(\tfrac{2s+1}{(s+1)(s+4)(s+5)}\right) G^*_{\mathrm{ZOH}}(s)\right]\right](z)\\ &= \left(\frac{z-1}{z}\right)\mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\tfrac{2s+1}{s(s+1)(s+4)(s+5)}\right]\right](z)\\ &= \left(\frac{z-1}{z}\right)\mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\tfrac{1/20}{s} + \tfrac{1/12}{s+1} + \tfrac{-7/12}{s+4} + \tfrac{9/20}{s+5}\right]\right](z)\\ &= \left(\frac{z-1}{z}\right)\left((1/20)\frac{z}{z-1} + (1/12)\frac{z}{z-e^{-T}} - (7/12)\frac{z}{z-e^{-4T}} + (9/20)\frac{z}{z-e^{-5T}}\right).\end{aligned}$$

For $T = 1\,\mathrm{s}$, we get

$$(HGG_{\mathrm{ZOH}})(z) = \frac{4.38z^2 - 2.435z - 0.0959}{60(z-0.368)(z-0.0183)(z-0.00674)}.$$

Since

$$E(z) = R(z) - (HGG_{\mathrm{ZOH}})(z)E(z),$$

we have

$$E(z) = \frac{1}{1+(HGG_{\mathrm{ZOH}})(z)}R(z).$$

The z-transform of the inverse laplace transform on the pseudo-sampled output gives

$$C(z) = \mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\left(\tfrac{2s+1}{(s+1)(s+2)(s+4)}\right) G^*_{\text{ZOH}}(s)\right]\right](z)E(z).$$

Again to simplify the notation, we let

$$\begin{aligned}(GG_{\text{ZOH}})(z) &= \mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\left(\tfrac{2s+1}{(s+1)(s+2)(s+4)}\right) G^*_{\text{ZOH}}(s)\right]\right](z)\\ &= \left(\frac{z-1}{z}\right)\mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\tfrac{2s+1}{s(s+1)(s+2)(s+4)}\right]\right](z)\\ &= \left(\frac{z-1}{z}\right)\mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\tfrac{1/8}{s}+\tfrac{1/3}{s+1}+\tfrac{-3/4}{s+2}+\tfrac{7/24}{s+4}\right]\right](z)\\ &= \left(\frac{z-1}{z}\right)\left((1/8)\frac{z}{z-1}+(1/3)\frac{z}{z-e^{-T}}-(3/4)\frac{z}{z-e^{-2T}}+(7/24)\frac{z}{z-e^{-4T}}\right).\end{aligned}$$

For $T = 1\,\text{s}$, we get

$$(GG_{\text{ZOH}})(z) = \frac{3.635z^2 - 1.776z - 0.25}{24(z-0.368)(z-0.135)(z-0.0183)}.$$

The output expression is

$$C(z) = (GG_{\text{ZOH}})(z)E(z) = \frac{(GG_{\text{ZOH}})(z)}{1+(HGG_{\text{ZOH}})(z)}R(z).$$

Substituting the expressions for $(GG_{\text{ZOH}})(z)$ and $(HGG_{\text{ZOH}})(z)$, we get

$$\frac{C(z)}{R(z)} = \frac{5(z-0.00674)\left(3.635z^2-1.776z-0.25\right)}{2(z-0.135)\left(60(z-0.368)(z-0.0183)(z-0.00674)+4.38z^2-2.435z-0.0959\right)}.$$

2. Consider a negative feedback discrete-time control system, where the loop gain is given by

$$G(z)H(z) = K\frac{z+0.5}{10z^3+4z^2-8.4z+1.44}.$$

Determine the range of stability in terms of the gain $K$.

**Solution:** For $G(z)H(z) = K\left((z+0.5)/(10z^3+4z^2-8.4z+1.44)\right)$, the characteristic equation is

$$1 + K\frac{z+0.5}{10z^3+4z^2-8.4z+1.44} = 0,$$

or

$$\frac{10z^3+4z^2-8.4z+1.44+K(z+0.5)}{10z^3+4z^2-8.4z+1.44} = 0..$$

Therefore the characteristic polynomial is

$$q(z) = 10z^3 + 4z^2 + (K-8.4)z + (0.5K+1.44).$$

To determine the range of stability for all $K$, we can use Jury's stability test criteria. In our case, the order of the system $n = 3$. The two boundary conditions are

$$q(1) > 0,$$

$$10(1)^3 + 4(1)^2 + (K - 8.4)(1) + (0.5K + 1.44) > 0,$$
$$K > -4.6933, \tag{2.1}$$

and

$$(-1)^n q(-1) > 0,$$
$$(-1)^3\big(10(-1)^3 + 4(-1)^2 + (K - 8.4)(-1) + (0.5K + 1.44)\big) > 0,$$
$$K > 7.68. \tag{2.2}$$

The pole-product condition is

$$|a_0| < a_n,$$
$$|0.5K + 1.44| < 10,$$
$$-10 < 0.5K + 1.44 < 10,$$
$$-11.44 < 0.5K < 8.56,$$
$$-22.88 < K < 17.12. \tag{2.3}$$

The rest of the conditions is to be obtained from the Jury's table.

| $z^0$ | $z^1$ | $z^2$ | $z^3$ |
|---|---|---|---|
| $a_0 = 0.5K + 1.44$ | $a_1 = K - 8.4$ | $a_2 = 4$ | $a_3 = 10$ |
| $a_3 = 10$ | $a_2 = 4$ | $a_1 = K - 8.4$ | $a_0 = 0.5K + 1.44$ |
| $a_0^1 = \det\begin{bmatrix} a_0 & a_3 \\ a_3 & a_0 \end{bmatrix}$ | $a_1^1 = \det\begin{bmatrix} a_0 & a_2 \\ a_3 & a_1 \end{bmatrix}$ | $a_2^1 = \det\begin{bmatrix} a_0 & a_1 \\ a_3 & a_2 \end{bmatrix}$ | |
| $= \det\begin{bmatrix} 0.5K + 1.44 & 10 \\ 10 & 0.5K + 1.44 \end{bmatrix}$ | | $= \det\begin{bmatrix} 0.5K + 1.44 & K - 8.4 \\ 10 & 4 \end{bmatrix}$ | |
| $a_0^1 = (0.5K + 1.44)^2 - 100$ | | $a_2^1 = -8K + 89.76$ | |

Since we have a third-order system, the table will only give one more additional condition.

$$|a_0^1| > |a_{n-1}^1|,$$
$$|(0.5K + 1.44)^2 - 100| > |-8K + 89.76|,$$
$$0.25|(K + 22.88)(K - 17.12)| > |8K - 89.76|.$$

From the Inequality (2.3), we know that $-22.88 < K < 17.12$, therefore we have

$$-0.25(K + 22.88)(K - 17.12) > |8K - 89.76| > 0.$$

**$-0.25(K + 22.88)(K - 17.12) > -(8K - 89.76) > 0$ Case:**

From the first portion, we get

$$-0.25K^2 + 6.56K + 8.1664 > 0,$$
$$-0.25(K + 1.19084)(K - 27.4308) > 0,$$
$$-1.19084 < K < 27.4308.$$

From the second portion, we get

$$-(8K - 89.76) > 0,$$

$$K < 11.22.$$

The intersection of the two regions gives

$$-1.19084 < K < 11.22. \tag{2.4}$$

**$-0.25(K + 22.88)(K - 17.12) > 8K - 89.76 > 0$ Case:**

From the first portion, we get

$$-0.25K^2 - 9.44K + 187.686 > 0,$$

$$-0.25(K + 52.1546)(K - 14.3946) > 0,$$

$$-52.1546 < K < 14.3946.$$

From the second portion, we get

$$8K - 89.76 > 0,$$

$$K > 11.22.$$

The intersection of the two regions gives

$$11.22 < K < 14.3946. \tag{2.5}$$

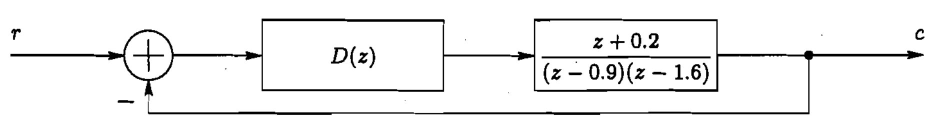
Since the two cases represent the same condition, we take the union of the regions described by Inequalities (2.4) and (2.5) to get

$$-1.19084 < K < 14.3946. \tag{2.6}$$

From the intersection of the regions described by Inequalities (2.1)–(2.3) and (2.6), we conclude that the system will be asymptotically stable, when

$$7.68 < K < 14.3946.$$

3. Consider the following system with a sampling period of 1/2 second.

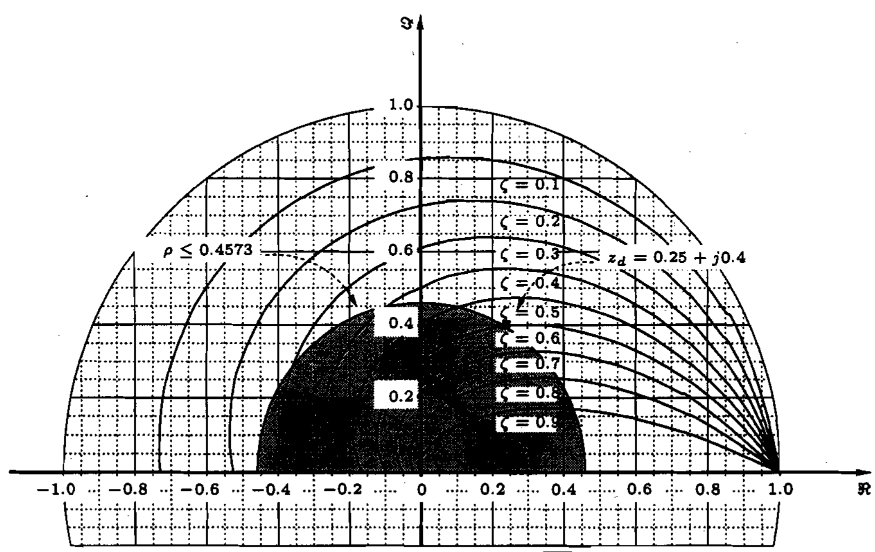



Design the simplest controller $D(z)$, such that the 2% settling-time is about 3 seconds, and the maximum percent-overshoot is approximately 10% for the unit-step input.

**Solution:** We determine the restrictions on the location of the desired-pole locations from the performance specifications.

| Given Requirements | General System Restrictions | Specific System Restrictions |
|---|---|---|
| Maximum percent-overshoot for the unit-step input | $M_p \approx 10\%$. | From the $\alpha$-$M_p$ curves, $\zeta = 0.6$ provides the broadest range of $\alpha$ values. |
| Settling time for the unit-step input | $\rho \approx (0.02)^{1/(k_{2\%s}-1)}$. | For $t_{2\%s} = k_{2\%s}T \leq 3\,\text{s}$, and $k_{2\%s} \leq 3/0.5 = 6$, when $T = 0.5\,\text{s}$; $\rho \approx (0.02)^{1/(6-1)} = 0.4573$. |

When we mark these restrictions on the z-plane, we determine that a possible set of desired-pole locations is at $z_d \approx 0.25 \pm j0.4$.




$$z = \Re(z) + j\Im(z) = e^{-\zeta\omega T} e^{j\sqrt{1-\zeta^2}\omega T}$$

The deficiency angle, $\phi$, needed at the desired location to ensure that one of the root-locus branches goes through the location, can be determined from the angular condition.

$$\phi + \measuredangle\big(z_d - (-0.2)\big) - \measuredangle\big(z_d - (0.9)\big) - \measuredangle\big(z_d - (1.6)\big) = (2k+1)\pi,$$

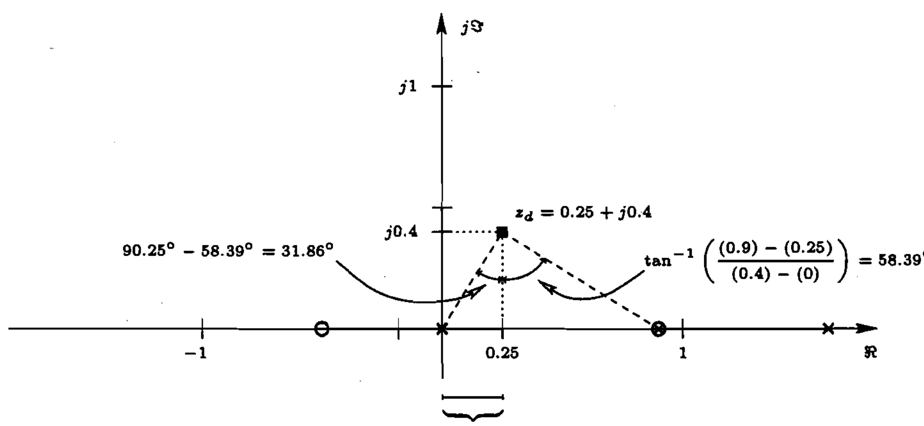
for an integer $k$. For $z_d = 0.25 + j0.4$,

$$\phi + \tan^{-1}\left(\frac{(0.4)-(0)}{(0.25)-(-0.2)}\right) - \tan^{-1}\left(\frac{(0.4)-(0)}{(0.25)-(0.9)}\right) - \tan^{-1}\left(\frac{(0.4)-(0)}{(0.25)-(1.6)}\right) = 180^\circ + k360^\circ,$$

$$\phi + 41.63^\circ - 148.39^\circ - 163.50^\circ = 180^\circ + k360^\circ,$$

or $\phi = 90.25^\circ$.

In order to preserve the system order so that transient specifications stay accurate, we need to cancel a pole or zero and place another one in such a way that the pole-zero combination provides the necessary deficiency angle at $z_d$. The best choice for cancellation is the pole at 0.9, since the other pole is an unstable pole, and we cannot realistically cancel an unstable pole.




From the above analysis,

$$D(z) = K\frac{z-0.9}{z}.$$

And the magnitude $K$ is obtained from the magnitude condition at $z_d$.

$$\left|D(z)G(z)\right|_{z=z_d} = 1,$$

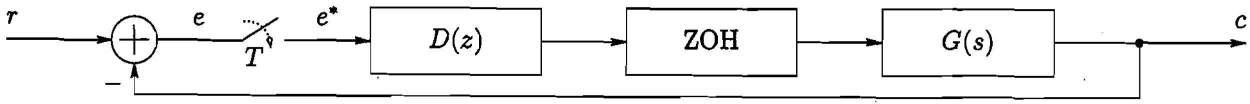
$$\left|K\frac{z+0.2}{z(z-1.6)}\right|_{z=0.25+j0.4} = 1,$$

or $K = 1.1031$. Therefore,

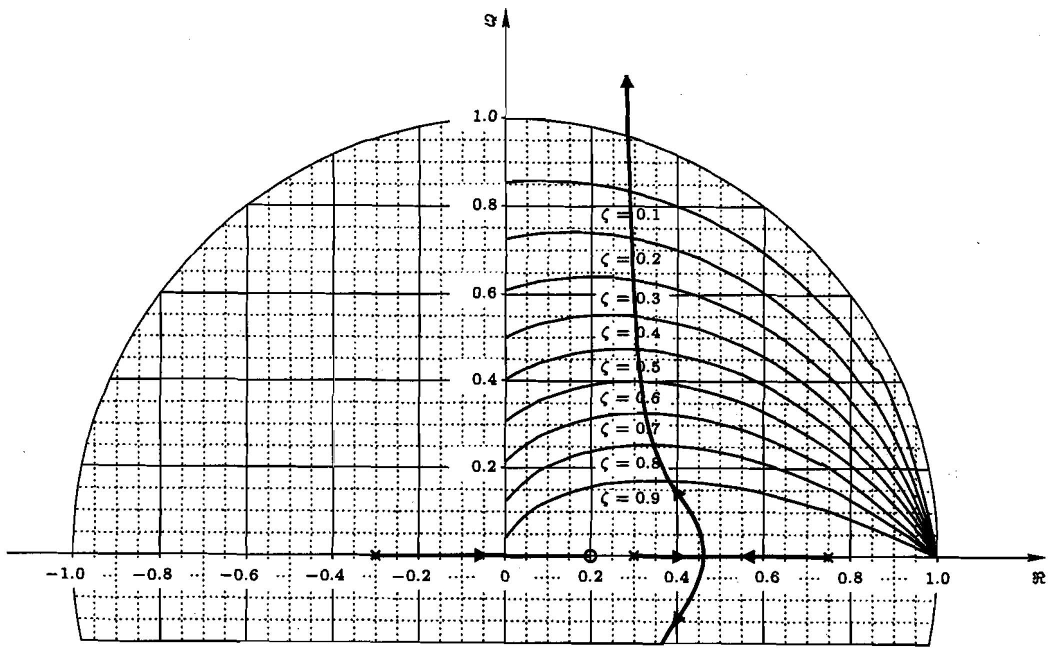
$$D(z) = 1.1031\frac{z-0.9}{z}$$

is one possible controller.

4. Consider the following control system. when $D(z) = K$.




The root-locus diagram for the discrete-time representation of the system is given below, when $D(z) = K$.




$$z = \Re(z) + j\Im(z) = e^{-\zeta\omega T} e^{j\sqrt{1-\zeta^2}\omega T}$$

(a) Design the simplest controller, such that the dominant closed-loop poles have $\zeta = 0.4$.

**Solution:** As we observe from the root-locus diagram, there are three open-loop poles and one finite open-loop zero. Since one of the branches of the diagram crosses the $\zeta = 0.4$ line, we can achieve the $\zeta = 0.4$ requirement by choosing an appropriate value for $K$. Indeed, since the other branch goes from $z = -0.3$ to 0.2, the branch crossing the $\zeta = 0.4$ line designates the dominant poles.

The intersection of the root-locus branch and the $\zeta = 0.4$ line is approximately at $z_d = 0.3+j0.55$, as we can observe from the root-locus diagram. The gain $K$ can be obtained from the magnitude condition. Directly observing the poles and the zeros of the open-loop gain, we have

$$(GG_{\text{ZOH}})(z)D(z) = \frac{K(z-0.2)}{(z+0.3)(z-0.3)(z-0.75)}.$$

The magnitude condition at $z_d = 0.3 + j0.55$,

$$\left|(GG_{\text{ZOH}})(z)D(z)\right|_{z=0.3+j0.55} = \left|\frac{K(z-0.2)}{(z+0.3)(z-0.3)(z-0.75)}\right|_{z=0.3+j0.55} = 1,$$

gives $K = 0.5691$. Therefore, the simplest controller is

$$D(z) = 0.5691.$$

(b) Specify the resulting $C(z)/R(z)$. All the unknown quantities must be evaluated, and each pole must be specified individually.

**Solution:** For $D(z) = 0.5691$, the open-loop gain is

$$(GG_{\text{ZOH}})(z)D(z) = \frac{0.5691(z-0.2)}{(z+0.3)(z-0.3)(z-0.75)}.$$

Since the discrete-time transfer function of the system is

$$\frac{C(z)}{R(z)} = \frac{(GG_{\text{ZOH}})(z)D(z)}{1+(GG_{\text{ZOH}})(z)D(z)},$$

we get

$$\frac{C(z)}{R(z)} = \frac{0.5691(z-0.2)}{(z+0.3)(z-0.3)(z-0.75)+0.5691(z-0.2)} = \frac{0.5691(z-0.2)}{z^3-0.75z^2+0.4791z-0.04632}.$$

Factoring the denominator to observe the closed-loop poles of the system, we get

$$\frac{C(z)}{R(z)} = \frac{0.5691(z-0.2)}{\left(z-0.118\right)\left(z-(0.3+j0.55)\right)\left(z-(0.3-j0.55)\right)}.$$