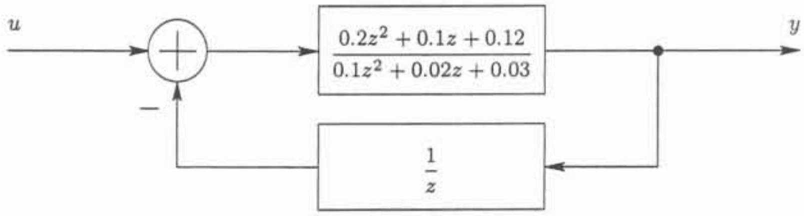
EE 331 **Exam#2** Apr. 25, 2006

75 minutes

1. The block diagram of a control system is given in the following figure.



Obtain a state-space representation of the system without any block-diagram reduction. (20pts)

2. A discrete-time linear control system is described by

$$x(k+1) = \begin{bmatrix} 0.5 & 0.1 \\ 0.6 & 0.4 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k),$$

$$y(k) = \begin{bmatrix} 1 & -0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \end{bmatrix} u(k),$$

where u, x, and y are the input, the state, and the output variables, respectively.

(a) Determine the transfer function of the system. (10pts)

(b) Determine the control sequence that would achieve the state $\begin{bmatrix} 18 & 32 \end{bmatrix}^T$ from the initial state $\begin{bmatrix} 100 & 100 \end{bmatrix}^T$, if such a control sequence exists. (20pts)

3. A discrete-time linear control system is described by

$$x(k+1) = \begin{bmatrix} 0.5 & 0.1 \\ 0.6 & 0.4 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k),$$

$$y(k) = \begin{bmatrix} 1 & -0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \end{bmatrix} u(k),$$

where u, x, and y are the input, the state, and the output variables, respectively. Design a full state-feedback controller, such that the 5% settling-time is reached within 3 sampling periods. HINT: Observing the location of the original poles may help in the possible choices for the new ones. (25pts)

4. Consider a linear control system described by the difference equation

$$x(k+1) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.7 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k),$$

where u and x are the input and the state variables, respectively. Determine the optimal control action $u(k)$ for $k \geq 0$ that would minimize the cost function

$$J = \sum_{k=0}^{\infty} \frac{1}{2} \left(x^T(k) \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} x(k) + 5u^2(k) \right),$$

when $x(0) = \begin{bmatrix} 10 & 20 \end{bmatrix}^T$. (25pts)

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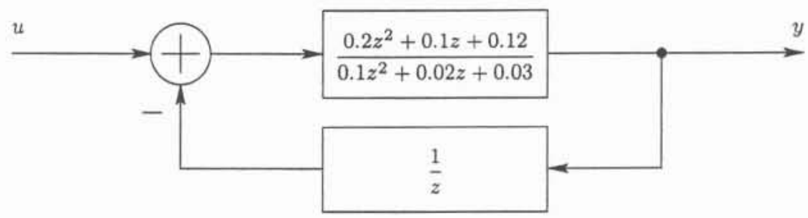
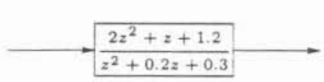
Solutions

1. The block diagram of a control system is given in the following figure.

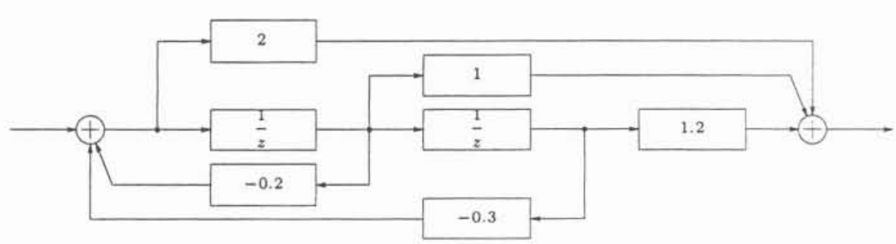


Obtain a state-space representation of the system without any block-diagram reduction.

Solution: In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.

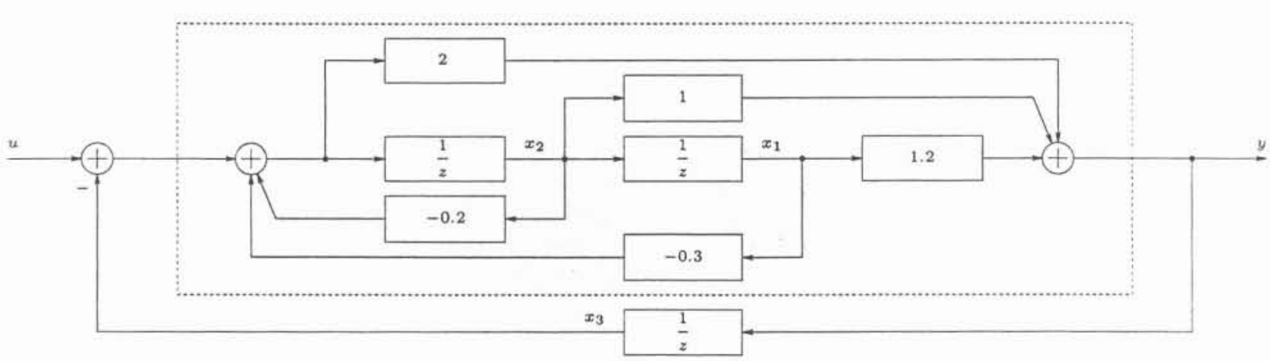


(a) The feedforward block.



(b) Controller realization form.

The connected and "expanded" block diagram is shown below.



After assigning the state variables as shown in the figure, we obtain

$$
\begin{aligned}
x_1(k+1) &= x_2(k),\\
x_2(k+1) &= -0.3x_1(k) - 0.2x_2(k) + \big(u(k) - x_3(k)\big) = -0.3x_1(k) - 0.2x_2(k) - x_3(k) + u(k),\\
x_3(k+1) &= y(k) = 1.2x_1(k) + x_2(k) + 2x_2(k+1)\\
&= 1.2x_1(k) + x_2(k) + 2\big(-0.3x_1(k) - 0.2x_2(k) - x_3(k) + u(k)\big)\\
&= 0.6x_1(k) + 0.6x_2(k) - 2x_3(k) + 2u(k),
\end{aligned}
$$

and

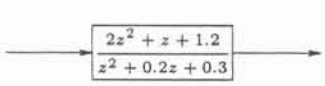
$$y(k) = 0.6x_1(k) + 0.6x_2(k) - 2x_3(k) + 2u(k).$$

Rewriting the equations in matrix form, we get

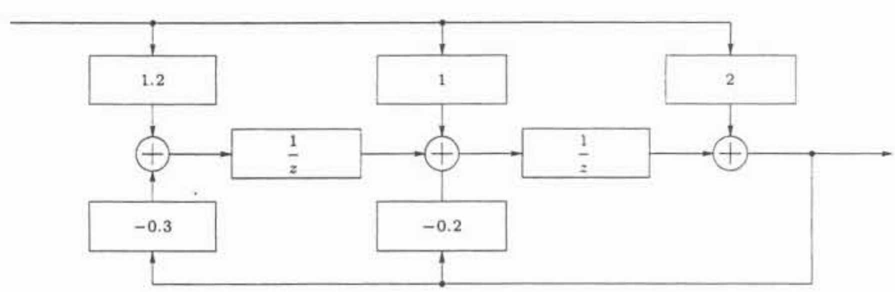
$$
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -0.3 & -0.2 & -1 \\ 0.6 & 0.6 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} u(k),
$$

$$
y(k) = \begin{bmatrix} 0.6 & 0.6 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 2 \end{bmatrix} u(k).
$$

If we use the observer realization form for each of the blocks, then we obtain a different state-space representation.

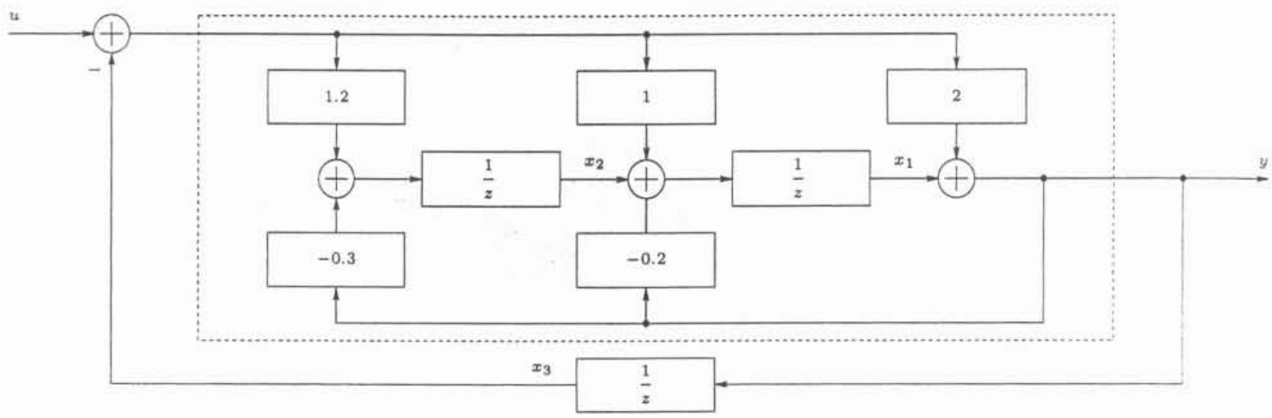


(a) The feedforward block.



(b) Observer realization form.

The connected and "expanded" block diagram is shown below.



Similarly, we obtain

$$x_1(k+1) = x_2(k) + \big(u(k) - x_3(k)\big) - 0.2y(k),$$
$$x_2(k+1) = 1.2\big(u(k) - x_3(k)\big) - 0.3y(k),$$
$$x_3(k+1) = y(k),$$

and

$$y(k) = x_1(k) + 2\big(u(k) - x_3(k)\big).$$

After substituting the output expression into the state-transition equations, we obtain

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -0.2 & 1 & -0.6 \\ -0.3 & 0 & -0.6 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0.6 \\ 0.6 \\ 2 \end{bmatrix} u(k),$$

$$y(k) = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 2 \end{bmatrix} u(k).$$

2. A discrete-time linear control system is described by

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 0.5 & 0.1 \\ 0.6 & 0.4 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k),$$
$$y(k) = \begin{bmatrix} 1 & -0.5 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 1 \end{bmatrix} u(k),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively.

(a) Determine the transfer function of the system.

Solution: The transfer matrix or the transfer function in the case of a single-input single-output control system described in the state-state representation

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k),$$
$$\boldsymbol{y}(k) = C\boldsymbol{x}(k) + D\boldsymbol{u}(k),$$

is

$$F(z) = C(zI - A)^{-1}B + D,$$

where

$$A = \begin{bmatrix} 0.5 & 0.1 \\ 0.6 & 0.4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & -0.5 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 \end{bmatrix},$$

and I is the appropriately dimensioned identity matrix. So,

$$\begin{aligned}
F(s) &= \begin{bmatrix} 1 & -0.5 \end{bmatrix} \left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.5 & 0.1 \\ 0.6 & 0.4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -0.5 \end{bmatrix} \begin{bmatrix} z-0.5 & -0.1 \\ -0.6 & z-0.4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \\
&= \frac{1}{(z-0.5)(z-0.4)-(-0.6)(-0.1)} \begin{bmatrix} 1 & -0.5 \end{bmatrix} \begin{bmatrix} z-0.4 & 0.1 \\ 0.6 & z-0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \\
&= \frac{1}{z^2 - 0.9z + 0.14} \begin{bmatrix} 1 & -0.5 \end{bmatrix} \begin{bmatrix} z-0.2 \\ 2z-0.4 \end{bmatrix} + 1 \\
&= \frac{1}{z^2 - 0.9z + 0.14} \left((z-0.2) - (z-0.2) \right) + 1 = 0 + 1.
\end{aligned}$$

Therefore, the transfer matrix is

$$F(s) = 1.$$

(b) Determine the control sequence that would achieve the state $\begin{bmatrix} 18 & 32 \end{bmatrix}^T$ from the initial state $\begin{bmatrix} 100 & 100 \end{bmatrix}^T$, if such a control sequence exists.

Solution: The property of being able to reach to any arbitrary final state from any other initial state is called reachability. One method to check the reachability of the system is by checking the rank of the controllability matrix

$$\mathcal{C}(A, B) = \begin{bmatrix} B \mid AB \mid \cdots \mid A^{n-1}B \end{bmatrix}.$$

In our case, the system order $n = 2$, and

$$\mathcal{C}(A, B) = \begin{bmatrix} B \mid AB \end{bmatrix} = \left[\begin{array}{c|c} 1 & 0.7 \\ 2 & 1.4 \end{array} \right].$$

In our system, we need to have 2 linearly independent rows or columns of $\mathcal{C}$ for reachability. Since the determinant of the controllability matrix is zero, and there is a non-zero element in the matrix; we conclude that the system is not reachable. However, even though the system is not reachable, some special states can be achieved from some other initial states. In order to check for this possibility, we need to determine whether or not there is a control sequence that would achieve the final state.

To determine the existence of the control sequence that would achieve the state $\begin{bmatrix} 18 & 32 \end{bmatrix}^T$ from the initial state $\begin{bmatrix} 100 & 100 \end{bmatrix}^T$, we may use a formula that can be derived by repeatedly applying the state-space equation.

$$\boldsymbol{x}(n) - A^n \boldsymbol{x}(0) = \mathcal{C}(A, B) \begin{bmatrix} \boldsymbol{u}(n-1) \\ \vdots \\ \boldsymbol{u}(0) \end{bmatrix}$$

for an nth order discrete-time system described by

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\boldsymbol{u}(k), \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) + D\boldsymbol{u}(k), \end{aligned}$$

where $\boldsymbol{u}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ are the input, the state, and the output variables, respectively. In our case, $n = 2$, and

$$\begin{aligned} \begin{bmatrix} 18 \\ 32 \end{bmatrix} - \begin{bmatrix} 0.5 & 0.1 \\ 0.6 & 0.4 \end{bmatrix}^2 \begin{bmatrix} 100 \\ 100 \end{bmatrix} &= \left[\begin{array}{c|c} 1 & 0.7 \\ 2 & 1.4 \end{array}\right] \left[\begin{array}{c} u(1) \\ \hline u(0) \end{array}\right], \\ \begin{bmatrix} 18 \\ 32 \end{bmatrix} - \begin{bmatrix} 0.31 & 0.09 \\ 0.54 & 0.22 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} &= \begin{bmatrix} 1 & 0.7 \\ 2 & 1.4 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}, \\ \begin{bmatrix} -22 \\ -44 \end{bmatrix} &= \begin{bmatrix} 1 & 0.7 \\ 2 & 1.4 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}. \end{aligned}$$

The above matrix equation provides two individual equations in $u(0)$ and $u(1)$. Writing these equations individually, we get

$$\begin{aligned} -22 &= u(1) + 0.7u(0), \\ -44 &= 2u(1) + 1.4u(0). \end{aligned}$$

Since these two equations represent the same relationship; in this case we do have a simultaneous solution, such that the control sequence

$$\begin{aligned} u(0) &= \alpha, \\ u(1) &= -0.7\alpha - 22 \end{aligned}$$

for any real number α would achieve the desired behavior in 2 steps.

3. A discrete-time linear control system is described by

$$\begin{aligned} \boldsymbol{x}(k+1) &= \begin{bmatrix} 0.5 & 0.1 \\ 0.6 & 0.4 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k), \\ y(k) &= \begin{bmatrix} 1 & -0.5 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 1 \end{bmatrix} u(k), \end{aligned}$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively. Design a full state-feedback controller, such that the 5% settling-time is reached within 3 sampling periods. HINT: Observing the location of the original poles may help in the possible choices for the new ones.

Solution: Settling time for the unit-step input implies

$$\rho \le (0.05)^{1/(k_{5\%s}-1)},$$

where ρ is the largest magnitude of the poles. For $t_{5\%s} = k_{5\%s}T \le 3T$, and $k_{5\%s} \le 3$;

$$\rho \le (0.05)^{1/(3-1)} = 0.2236.$$

In other words, the magnitudes of the closed-loop poles need to be less than 0.2236. However, since the system is not reachable, we won't be able to set all the poles arbitrarily by state-feedback control. Some of the original poles will be fixed. To determine which poles will be fixed, we first compute the original poles from the original characteristic polynomial, where

$$q(z) = \det(zI - A) = \det\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.5 & 0.1 \\ 0.6 & 0.4 \end{bmatrix}\right)$$

$$= z^2 - 0.9z + 0.14 = (z-0.2)(z-0.7).$$

In other words, the original poles are at 0.2 and 0.7.

The characteristic polynomial q_c under state-feedback gain $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, such that the input $u = K\boldsymbol{x}$, can be determined from

$$\begin{aligned}
q_c(z) &= \det\big(zI - (A+BK)\big) \\
&= \det\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} 0.5 & 0.1 \\ 0.6 & 0.4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right)\right) \\
&= \det\begin{bmatrix} z-0.5-k_1 & -0.1-k_2 \\ -0.6-2k_1 & z-0.4-2k_2 \end{bmatrix} \\
&= z^2 + (-k_1 - 2k_2 - 0.9)z + (0.2k_1 + 0.4k_2 + 0.14).
\end{aligned}$$

Considering that the desired closed-loop system poles are at p_1 and p_2, the closed-loop desired characteristic polynomial is

$$q_{c_d}(z) = (z-p_1)(z-p_2) = z^2 + (-p_1 - p_2)z + p_1p_2.$$

Setting $q_c(z) = q_{c_d}(z)$, we get

$$-k_1 - 2k_2 - 0.9 = -p_1 - p_2,$$

or

$$k_1 + 2k_2 = p_1 + p_2 - 0.9;$$

and

$$0.2k_1 + 0.4k_2 + 0.14 = p_1p_2,$$

or

$$k_1 + 2k_2 = 5p_1p_2 - 0.7.$$

The only way these two equations would have a solution is if

$$p_1 + p_2 - 0.9 = 5p_1p_2 - 0.7,$$

or

$$p_2 = \frac{p_1 - 0.2}{5p_1 - 1} = 0.2\left(\frac{p_1 - 0.2}{p_1 - 0.2}\right).$$

From the above equation, we observe that if $p_1 \neq 0.2$, then $p_2 = 0.2$. In other words, the pole at 0.2 is fixed, and it can't be changed. Note here that we could have found about this fixed pole by trial and error as well.

In our case, since the poles have to have magnitudes less than 0.2236, the fixed pole at 0.2 doesn't prevent us to design a controller satisfying the requirement. The control gain equations

$$\begin{aligned} k_1 + 2k_2 &= p_1 + p_2 - 0.9 \\ k_1 + 2k_2 &= 5p_1p_2 - 0.7 \end{aligned}$$

become

$$\begin{aligned} k_1 + 2k_2 &= (0.2) + (0.1) - 0.9 = -0.6 \\ k_1 + 2k_2 &= 5(0.2)(0.1) - 0.7 = -0.6, \end{aligned}$$

if we assume that the desired closed-loop poles are at 0.2 and 0.1. Solving the above equation for k_1 in terms of $k_2 = \alpha$, we conclude that one possible state-feedback control is

$$u(k) = \begin{bmatrix} -(2\alpha + 0.6) & \alpha \end{bmatrix} \boldsymbol{x}(k)$$

for $k \geq 0$ and for any real number α.

4. Consider a linear control system described by the difference equation

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.7 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k),$$

where u and $\boldsymbol{x}$ are the input and the state variables, respectively. Determine the optimal control action $u(k)$ for $k \geq 0$ that would minimize the cost function

$$J = \sum_{k=0}^{\infty} \frac{1}{2}\left(\boldsymbol{x}^T(k) \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \boldsymbol{x}(k) + 5u^2(k)\right),$$

when $\boldsymbol{x}(0) = \begin{bmatrix} 10 & 20 \end{bmatrix}^T$.

Solution: Since the infinite-time cost function is quadratic in the state and the input variables, the optimal control can be expressed in state-feedback form, such that

$$u(k) = -\left(R + B^T\bar{P}B\right)^{-1}B^T\bar{P}A\boldsymbol{x}(k),$$

and the positive definite symmetric matrix $\bar{P}$ satisfies the algebraic riccati equation

$$\bar{P} = Q + A^T\left(I + \bar{P}BR^{-1}B^T\right)^{-1}\bar{P}A$$

where I is the appropriately dimensioned identity matrix, and Q and R are from the cost function

$$J = \sum_{k=0}^{\infty} \frac{1}{2}\Big(\boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^T(k)R\boldsymbol{u}(k)\Big),$$

for the control system described by

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k).$$

In our case, since A, Q, and R are all diagonal matrices, the solution $\bar{P}$ to the algebraic riccati equation is also diagonal. So, assume

$$\bar{P} = \begin{bmatrix} \bar{p}_1 & 0 \\ 0 & \bar{p}_2 \end{bmatrix}.$$

Then, the algebraic riccati equation becomes

$$\begin{bmatrix} \bar{p}_1 & 0 \\ 0 & \bar{p}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.7 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \bar{p}_1 & 0 \\ 0 & \bar{p}_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(5\right)^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} \bar{p}_1 & 0 \\ 0 & \bar{p}_2 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.7 \end{bmatrix},$$

$$\begin{aligned} \begin{bmatrix} \bar{p}_1 & 0 \\ 0 & \bar{p}_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 + \bar{p}_2/5 \end{bmatrix}^{-1} \begin{bmatrix} 0.5\bar{p}_1 & 0 \\ 0 & 0.7\bar{p}_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} + \begin{bmatrix} 0.25\bar{p}_1 & 0 \\ 0 & 0.49\bar{p}_2/(1+\bar{p}_2/5) \end{bmatrix}. \end{aligned}$$

From the above equation, we get

$$\bar{p}_1 = 1 + 0.25\bar{p}_1,$$

or

$$\bar{p}_1 = 1.3333;$$

and

$$\bar{p}_2 = 10 + \frac{0.49\bar{p}_2}{1+\bar{p}_2/5},$$

or

$$\bar{p}_2{}^2/5 - 1.49\bar{p}_2 - 10 = (1/5)(\bar{p}_2 + 4.2672)(\bar{p}_2 - 11.7172) = 0.$$

Choosing the positive definite solution for $\bar{P}$, we get

$$\bar{P} = \begin{bmatrix} 1.3333 & 0 \\ 0 & 11.7172 \end{bmatrix}.$$

Therefore, the optimal control is

$$\begin{aligned} u(k) &= -\left(R + B^T\bar{P}B\right)^{-1} B^T \bar{P} A \boldsymbol{x}(k) \\ &= -\left(5 + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1.3333 & 0 \\ 0 & 11.7172 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1.3333 & 0 \\ 0 & 11.7172 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.7 \end{bmatrix} \boldsymbol{x}(k) \\ &= -\frac{1}{16.7172} \begin{bmatrix} 0 & 11.7172 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.7 \end{bmatrix} \boldsymbol{x}(k) \end{aligned}$$

or

$$u(k) = -\begin{bmatrix} 0 & 0.49 \end{bmatrix} \boldsymbol{x}(k) \text{ for } k \geq 0.$$