EE 331 **Exam#2** **Apr. 24, 2008**

75 minutes

1. A discrete-time control system is described by

$$Y(z) = \frac{z-0.3}{z^2-0.2z+0.05}\,U_1(z) + \frac{z^2-0.01}{z^2-0.2z+0.05}\,U_2(z),$$

where $U_1 = \mathcal{Z}[\,u_1\,]$, $U_2 = \mathcal{Z}[\,u_2\,]$, and $Y = \mathcal{Z}[\,y\,]$ are the two input and the one output variables, respectively. Obtain a state-space representation of the system with minimal number of state variables. (20pts)

2. A continuous-time control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively. Determine its discrete-time state-space representation, when $T = 0.5\,\mathrm{s}$. (25pts)

3. A discrete-time linear control system is described by

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -0.48 & -1.4 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k),$$
$$y(k) = \begin{bmatrix} 0.6 & 1 \end{bmatrix} \boldsymbol{x}(k),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively.

(a) Determine the transfer function $Y(z)/U(z)$ of the system, where $U = \mathcal{Z}[\,u\,]$ and $Y = \mathcal{Z}[\,y\,]$ (10pts)

(b) Design a full state-feedback controller, such that the closed-system system poles are located at $z = 0.3 \pm j0.2$. (15pts)

(c) Implement the controller of the previous part by assuming that only the output is available. (15pts)

(d) The output is observed to be $y(k) = 2(-0.8)^k$ for $k \geq 0$, when $u(k) = 0$. Obtain all the instances of the state variable $\boldsymbol{x}(k)$ that can be determined. (15pts)

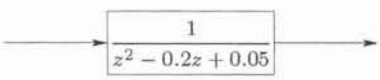
Solutions

1. A discrete-time control system is described by

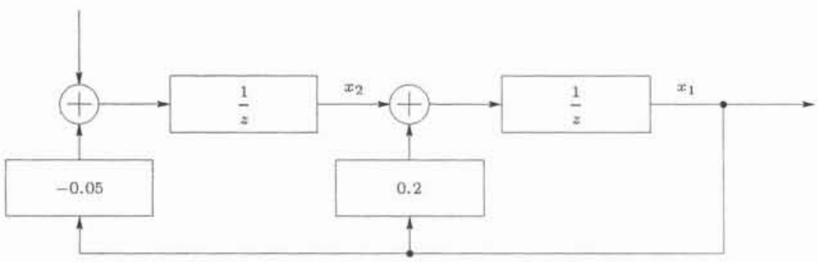
$$Y(z) = \frac{z - 0.3}{z^2 - 0.2z + 0.05}\, U_1(z) + \frac{z^2 - 0.01}{z^2 - 0.2z + 0.05}\, U_2(z),$$

where $U_1 = \mathcal{Z}[\, u_1 \,]$, $U_2 = \mathcal{Z}[\, u_2 \,]$, and $Y = \mathcal{Z}[\, y \,]$ are the two input and the one output variables, respectively. Obtain a state-space representation of the system with minimal number of state variables.

Solution: From the transfer matrix, we observe that there is only one output to the system. As a result, we may be able to factor out the common denominator and realize the denominator polynomial in the observer realization form, and the two numerator polynomials can be generated independently as the feedforward terms.

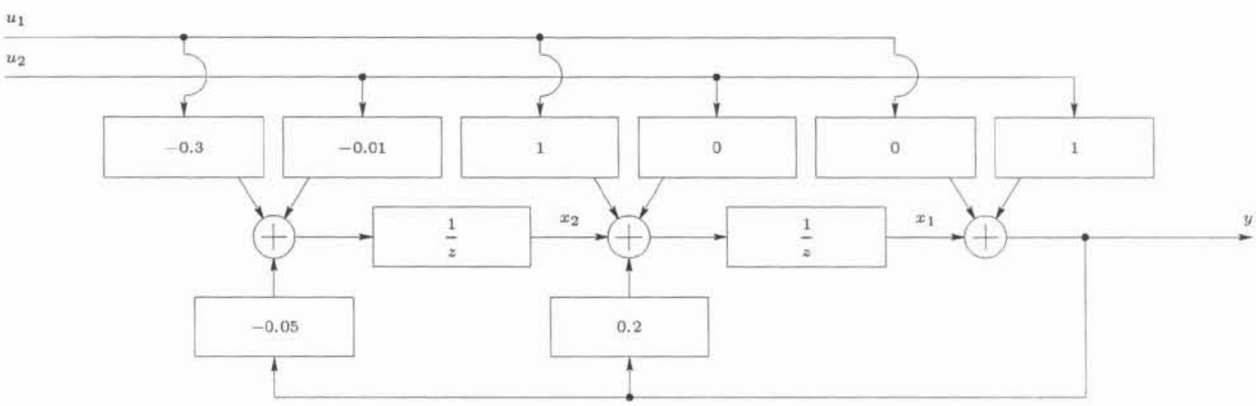


(a) The feedback portion.



(b) Observer realization form.

When the two elements in the numerator matrix are generated as the feedforward terms, we get



Next, we assign the state variables as shown in the figure and obtain

$$x_1(k+1) = x_2(k) + 0.2y(k) + u_1(k),$$

$$x_2(k+1) = -0.05y(k) - 0.3u_1(k) - 0.01u_2(k),$$

and

$$y(k) = x_1(k) + u_2(k).$$

We need to substitute the y expression into the state equations to obtain a state-space representation.

$$\begin{aligned}
x_1(k+1) &= x_2(k) + 0.2\big(x_1(k) + u_2(k)\big) + u_1(k) \\
&= 0.2x_1(k) + x_2(k) + u_1(k) + 0.2u_2(k), \\
x_2(k+1) &= -0.05\big(x_1(k) + u_2(k)\big) - 0.3u_1(k) - 0.01u_2(k) \\
&= -0.05x_1(k) - 0.3u_1(k) - 0.06u_2(k),
\end{aligned}$$

and

$$y(k) = x_1(k) + u_2(k).$$

After expressing the above equations in matrix form, we get the state-space representation

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.2 & 1 \\ -0.05 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 & 0.2 \\ -0.3 & -0.06 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix},$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}.$$

2. A continuous-time control system is described by

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(t),
\end{aligned}$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively. Determine its discrete-time state-space representation, when $T = 0.5\,\text{s}$.

Solution: The representation of a continuous-time state-representation described by

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}$$

is given by

$$\begin{aligned}
\boldsymbol{x}(k+1) &= \Phi(T,0)\boldsymbol{x}(k) + \left(\int_0^T \Phi(T,t)B\,\mathrm{d}t\right)\boldsymbol{u}(k) \\
\boldsymbol{y}(k) &= C\boldsymbol{x}(k) + D\boldsymbol{u}(k),
\end{aligned}$$

where the state transition matrix

$$\Phi(t,t_0) = e^{A(t-t_0)} = \mathcal{L}_s^{-1}\big[\,(sI - A)^{-1}\,\big](t - t_0).$$

In our case,

$$\Phi(t,0) = \mathcal{L}_s^{-1}\left[\,(sI - A)^{-1}\,\right](t) = \mathcal{L}_s^{-1}\left[\left[\begin{array}{cc} s & -1 \\ 2 & s+3 \end{array}\right]^{-1}\right](t)$$

$$= \mathcal{L}_s^{-1}\left[\begin{array}{cc} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{array}\right](t)$$

$$= \left[\begin{array}{cc} \mathcal{L}_s^{-1}\left[\,\frac{2}{s+1} - \frac{1}{s+2}\,\right](t) & \mathcal{L}_s^{-1}\left[\,\frac{1}{s+1} - \frac{1}{s+2}\,\right](t) \\ \mathcal{L}_s^{-1}\left[\,\frac{-2}{s+1} + \frac{2}{s+2}\,\right](t) & \mathcal{L}_s^{-1}\left[\,\frac{-1}{s+1} + \frac{2}{s+2}\,\right](t) \end{array}\right]$$

$$= \left[\begin{array}{cc} \left(2e^{-t} - e^{-2t}\right) & \left(e^{-t} - e^{-2t}\right) \\ \left(-2e^{-t} + 2e^{-2t}\right) & \left(-e^{-t} + 2e^{-2t}\right) \end{array}\right]$$

So,

$$A = \Phi(T,0) = \left[\begin{array}{cc} \left(2e^{-T} - e^{-2T}\right) & \left(e^{-T} - e^{-2T}\right) \\ \left(-2e^{-T} + 2e^{-2T}\right) & \left(-e^{-T} + 2e^{-2T}\right) \end{array}\right]_{T=0.5} = \left[\begin{array}{cc} 0.8452 & 0.2387 \\ -0.4773 & 0.1292 \end{array}\right].$$

Also,

$$\int_0^T \Phi(t,0)B\,\mathrm{d}t = \int_0^T \left[\begin{array}{cc} \left(2e^{-t} - e^{-2t}\right) & \left(e^{-t} - e^{-2t}\right) \\ \left(-2e^{-t} + 2e^{-2t}\right) & \left(-e^{-t} + 2e^{-2t}\right) \end{array}\right]\left[\begin{array}{c} 0 \\ 1 \end{array}\right]\mathrm{d}t$$

$$= \int_0^T \left[\begin{array}{c} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} \end{array}\right]\mathrm{d}t = \left[\begin{array}{c} -e^{-t} + (1/2)e^{-2t} \\ e^{-t} - e^{-2t} \end{array}\right]_{t=0}^{t=T}$$

$$= \left[\begin{array}{c} \left(-e^{-T} + (1/2)e^{-2T}\right) - \left(-1 + (1/2)\right) \\ \left(e^{-T} - e^{-2T}\right) - \left(1 - 1\right) \end{array}\right] = \left[\begin{array}{c} (1/2) - e^{-T} + (1/2)e^{-2T} \\ e^{-T} - e^{-2T} \end{array}\right].$$

As a result,

$$B = \left(\int_0^T \Phi(t,0)B\,\mathrm{d}t\right)_{T=0.5} = \left[\begin{array}{c} 0.0774 \\ 0.2387 \end{array}\right].$$

Therefore, the discrete-time state-space representation of the continuous-time system is

$$\boldsymbol{x}(k+1) = \left[\begin{array}{cc} 0.8452 & 0.2387 \\ -0.4773 & 0.1292 \end{array}\right]\boldsymbol{x}(k) + \left[\begin{array}{c} 0.0774 \\ 0.2387 \end{array}\right]\boldsymbol{u}(k)$$

$$\boldsymbol{y}(k) = \left[\begin{array}{cc} 1 & 0 \end{array}\right]\boldsymbol{x}(k).$$

3. A discrete-time linear control system is described by

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -0.48 & -1.4 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k),$$
$$y(k) = \begin{bmatrix} 0.6 & 1 \end{bmatrix} \boldsymbol{x}(k),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively.

(a) Determine the transfer function $Y(z)/U(z)$ of the system, where $U = \mathcal{Z}[\,u\,]$ and $Y = \mathcal{Z}[\,y\,]$

Solution: The transfer matrix or the transfer function in the case of a single-input single-output control system described in the state-state representation

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k),$$
$$\boldsymbol{y}(k) = C\boldsymbol{x}(k) + D\boldsymbol{u}(k),$$

is

$$F(z) = C(zI - A)^{-1}B + D,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -0.48 & -1.4 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.6 & 1 \end{bmatrix}, \qquad D = 0,$$

and I is the appropriately dimensioned identity matrix. So,

$$F(s) = \begin{bmatrix} 0.6 & 1 \end{bmatrix} \left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.48 & -1.4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 0.6 & 1 \end{bmatrix} \begin{bmatrix} z & -1 \\ 0.48 & z+1.4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{z(z+1.4) - (0.48)(-1)} \begin{bmatrix} 0.6 & 1 \end{bmatrix} \begin{bmatrix} z+1.4 & 1 \\ -0.48 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(z+0.8)(z+0.6)} \begin{bmatrix} 0.6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} = \frac{z+0.6}{(z+0.8)(z+0.6)}.$$

Therefore, the transfer function is

$$F(s) = \frac{1}{z+0.8}.$$

(b) Design a full state-feedback controller, such that the closed-system system poles are located at $z = 0.3 \pm j0.2$.

Solution: The characteristic polynomial q_c under state-feedback gain $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, such that the input $u = K\boldsymbol{x}$, can be determined from

$$\begin{aligned} q_c(z) &= \det\big(zI - (A + BK)\big) \\ &= \det\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ -0.48 & -1.4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right)\right) \\ &= \det\begin{bmatrix} z & -1 \\ -0.48 - k_1 & z + 1.4 - k_2 \end{bmatrix} \\ &= z^2 + (-k_2 + 1.4)z + (-k_1 + 0.48). \end{aligned}$$

Considering that the desired closed-loop system poles are at $z = 0.3 \pm j0.2$, the closed-loop desired characteristic polynomial is

$$q_{c_d}(z) = \big(z - (0.3 + j0.2)\big)\big(z - (0.3 + j0.2)\big) = z^2 - 0.6z + 0.13.$$

Setting $q_c(z) = q_{c_d}(z)$, we get

$$-k_2 + 1.4 = -0.6,$$

or

$$k_2 = 2;$$

and

$$-k_1 + 0.48 = 0.13,$$

or

$$k_1 = 0.35.$$

Therefore the state-feedback control is

$$u(k) = \begin{bmatrix} 0.35 & 2 \end{bmatrix} \boldsymbol{x}(k)$$

for $k \geq 0$.

(c) Implement the controller of the previous part by assuming that only the output is available.

Solution: When only the output is available, state-feedback control can still be implemented if an observer is used. Moreover, we know that if a system is observable, we can place the closed-loop poles of the observer at any desired location via error-feedback control. However, in this case, the system is reachable, and one of the original system poles cancel out in the transfer function. Therefore, the system cannot be observable, and we won't be able to place both of the observer poles at any desired location. From the value of the canceled pole, we may also conclude that the pole at $z = -0.6$ cannot be moved in an observer design. Under these restrictions and for $u(k) = K\boldsymbol{x}(k) + v(k)$, the desired observer-characteristic polynomial can only be

$$q_{o_d}(z) = (z + 0.6)(z - \alpha);$$

where $-1 < \alpha < 1$. So assuming that

$$\boldsymbol{e}(k) = L\big(\hat{y}(k) - y(k)\big) = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \big(\hat{y}(k) - y(k)\big)$$

for some observer-error gain matrix L, where $\hat{y}$ is the observer output variable; and letting $\alpha = 0.1$, the desired observer-characteristic polynomial becomes

$$q_{\mathrm{o_d}}(z) = (z+0.6)(z-0.1) = z^2 + 0.5z - 0.06.$$

The observer-characteristic polynomial q_{o} under the error feedback can be determined from the denominator of the transfer function of the observer, such that

$$\begin{aligned} q_{\mathrm{o}}(z) &= \det\big(zI - (A + LC)\big) \\ &= \det\left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ -0.48 & -1.4 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 0.6 & 1 \end{bmatrix} \right) \right) \\ &= z^2 + (-0.6l_1 - l_2 + 1.4)z + (-0.36l_1 - 0.6l_2 + 0.48). \end{aligned}$$

Setting $q_{\mathrm{o}}(z) = q_{\mathrm{o_d}}(z)$, we get

$$-0.6l_1 - l_2 + 1.4 = 0.5,$$

and

$$-0.36l_1 - 0.6l_2 + 0.48 = -0.06.$$

Obviously, the two equations are the same, and we get infinitely many solutions with one equation. Letting $l_1 = \alpha$ in

$$-0.6l_1 - l_2 = -0.9,$$

we get

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0.9 - 0.6\alpha \end{bmatrix},$$

or

$$\boldsymbol{e}(k) = \begin{bmatrix} \alpha \\ 0.9 - 0.6\alpha \end{bmatrix} (\hat{y}(k) - y(k))$$

for any α, where $\boldsymbol{e}$ and $\hat{y}$ are the error feedback and the observer output variables, respectively.

(d) The output is observed to be $y(k) = 2(-0.8)^k$ for $k \geq 0$, when $u(k) = 0$. Obtain all the instances of the state variable $\boldsymbol{x}(k)$ that can be determined.

Solution: In order to observe the initial conditions from the future values of the output, the system needs to be observable. The observability property can be checked by the rank of the observability matrix

$$\mathcal{O}(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

In our case, the observability matrix is

$$\mathcal{O}(C, A) = \begin{bmatrix} 0.6 & 1 \\ -0.48 & 0.8 \end{bmatrix},$$

and $\operatorname{rank}\big(\mathcal{O}(C, A)\big) = 1 < 2$.

As a result, we know that $\boldsymbol{x}(0)$ cannot be determined from $y(k)$ for $k \geq 0$. We also know that we can't determine $\boldsymbol{x}(-l)$ for $l > 0$ as well; since determining $\boldsymbol{x}(-l)$ implies determining $\boldsymbol{x}(0) = A^l\boldsymbol{x}(-l)$. Similarly, because of the invertibility of the state matrix A, we can't determine $\boldsymbol{x}(k)$ for $k > 0$; since determining $\boldsymbol{x}(k)$ implies determining $\boldsymbol{x}(0) = A^{-k}\boldsymbol{x}(k)$.

Therefore, there is no instance of the state variable that can be determined from the output.