EE 431 | **Exam#1** | **Oct. 18, 2005**

75 minutes

1. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Determine $\sin(A)$ and $\cos(A)$. Simplify the expressions as much as possible. (25pts)

HINT: Euler's identities,

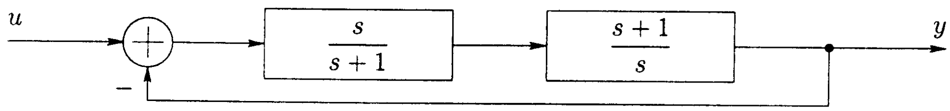
$$\sin(\alpha) = \frac{e^{j\alpha} - e^{-j\alpha}}{2j} \quad \text{and} \quad \cos(\alpha) = \frac{e^{j\alpha} + e^{-j\alpha}}{2},$$

as well as the hyperbolic function definitions,

$$\sinh(\alpha) = \frac{e^{\alpha} - e^{-\alpha}}{2} \quad \text{and} \quad \cosh(\alpha) = \frac{e^{\alpha} + e^{-\alpha}}{2},$$

may be used to simplify the final expressions.

2. The block diagram of a control system is given below.



(a) Obtain a state-space representation of the system without any block-diagram reduction. (15pts)

(b) Determine the transfer function of the system from its state-space representation. (10pts)

3. A time-varying linear control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -6 + 5e^{-t} & 3 - 3e^{-t} \\ -10 + 10e^{-t} & 5 - 6e^{-t} \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

where u and $\boldsymbol{x}$ are the input and the state variables, respectively. Determine $\boldsymbol{x}(t)$ for $t \geq 0$, when $\boldsymbol{x}(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, and $u(t) = 0$ for $t \geq 0$. (30pts)

4. A control system is described in state-space representation, such that

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t),$$

where $\boldsymbol{u}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ are the input, the state, and the output variables, respectively. For the following A, B, C, and D matrices, determine whether the system is asymptotically stable, marginally stable, or unstable; and whether it is bounded-input-bounded-output stable or not. Justify your answer.

(a)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and } D = 0.$$

(10pts)

(b)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \text{ and } D = 0.$$

(10pts)

EE 431 **Exam#1 Solutions** Oct. 18, 2005

1. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Determine $\sin(A)$ and $\cos(A)$. Simplify the expressions as much as possible.

HINT: Euler's identities,

$$\sin(\alpha) = \frac{e^{j\alpha} - e^{-j\alpha}}{2j} \quad \text{and} \quad \cos(\alpha) = \frac{e^{j\alpha} + e^{-j\alpha}}{2},$$

as well as the hyperbolic function definitions,

$$\sinh(\alpha) = \frac{e^{\alpha} - e^{-\alpha}}{2} \quad \text{and} \quad \cosh(\alpha) = \frac{e^{\alpha} + e^{-\alpha}}{2},$$

may be used to simplify the final expressions.

Solution: Any function, that has a non-trivial taylor series expansion, of an nth order matrix can be determined either by its taylor series expansion, where

$$f(A) = \sum_{i=0}^{\infty} \frac{1}{i!} \left[\frac{\mathrm{d}^i f(x)}{\mathrm{d}x^i}\right]_{x=0} A^i;$$

by the use of the Cayley-Hamilton's theorem, where

$$f(A) = \sum_{i=0}^{n-1} \alpha_i A^i$$

for some scalars α_i, $i = 0, \ldots, n-1$; or by diagonalization, where

$$f(A) = T f(T^{-1}AT) T^{-1}$$

for a transformation T, such that $T^{-1}AT$ is in jordan form and the evaluation of $f(T^{-1}AT)$ is directly performed.

In the use of the Cayley-Hamilton's theorem, the scalars are determined by the application of the eigenvectors to the above equation that results in the set of equations

$$f(\lambda_1) = \alpha_0 + \alpha_1\lambda_1 + \ldots + \alpha_{n-1}\lambda_1^{n-1}$$
$$\vdots$$
$$f(\lambda_n) = \alpha_0 + \alpha_1\lambda_n + \ldots + \alpha_{n-1}\lambda_n^{n-1},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues, and they are determined from

$$\det(\lambda I - A) = 0.$$

In our case, $n = 2$; so

$$f(A) = \alpha_0 I + \alpha_1 A;$$

and the eigenvalues are determined from

$$\det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \det \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} = \lambda^2 + 1 = 0.$$

or $\lambda_{1,2} = \pm j$. For $f = \sin$, the set of equations becomes

$$\sin(j) = \alpha_0 + \alpha_1(j),$$
$$\sin(-j) = \alpha_0 + \alpha_1(-j).$$

Observing that $\sin(-\theta) = -\sin(\theta)$ and solving the above set of equations simultaneously, we get

$$\alpha_0 = 0,$$

and

$$\alpha_1 = \frac{\sin(j)}{j} = \frac{(e^{-1} - e^{+1})/2j}{j} = \frac{e^{+1} - e^{-1}}{2} = \sinh(1).$$

As a result,

$$\sin(A) = \sinh(1)A = \sinh(1) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \approx \begin{bmatrix} 0 & 1.1752 \\ -1.1752 & 0 \end{bmatrix}.$$

For $f = \cos$; we get

$$\cos(j) = \alpha_0 + \alpha_1(j),$$
$$\cos(-j) = \alpha_0 + \alpha_1(-j).$$

This time $\cos(-\theta) = \cos(\theta)$, and we get

$$\alpha_0 = \cos(j) = \frac{(e^{-1} + e^{+1})}{2} = \cosh(1),$$

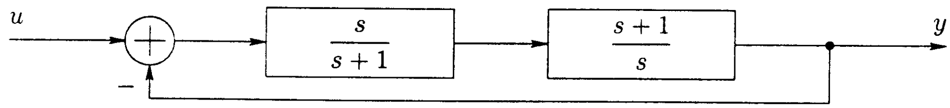
and

$$\alpha_1 = 0.$$

Similarly,

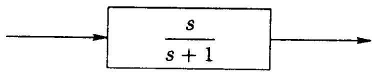
$$\cos(A) = \cosh(1)I = \cosh(1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1.5431 & 0 \\ 0 & 1.5431 \end{bmatrix}.$$

2. The block diagram of a control system is given below.

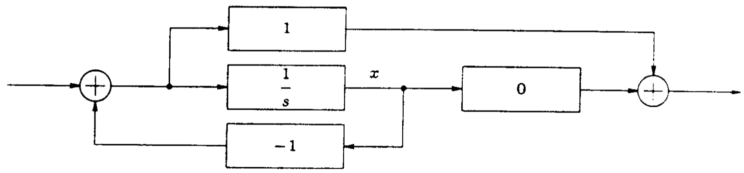


(a) Obtain a state-space representation of the system without any block-diagram reduction.

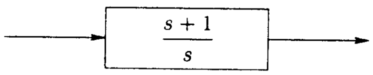
Solution: In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.



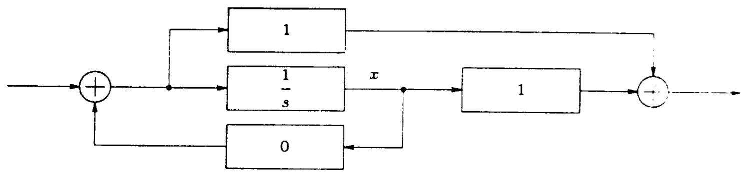
(a) The first feedforward gain block.



(b) Controller realization form.

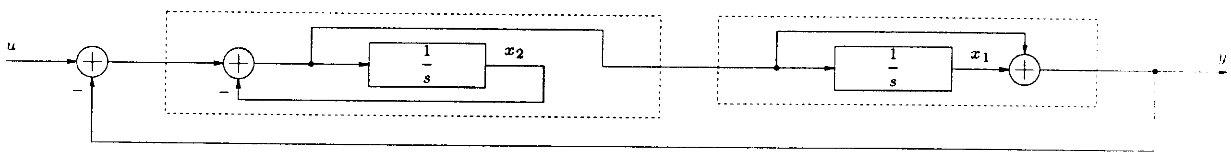


(a) The second feedforward gain block.



(b) Controller realization form.

The connected and "expanded" block diagram is shown below.



After assigning the state variables as shown in the figure, we obtain

$$\dot{x}_1 = \dot{x}_2,$$

$$\dot{x}_2 = -x_2 + (u - y),$$

and

$$y = x_1 + \dot{x}_1 = x_1 + \dot{x}_2.$$

Substituting the output equation to the second state-variable equation, we get

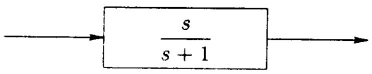
$$\begin{aligned} \dot{x}_2 &= -x_2 + u - x_1 - \dot{x}_2 \\ 2\dot{x}_2 &= -x_1 - x_2 + u \\ \dot{x}_2 &= -(1/2)x_1 - (1/2)x_2 + (1/2)u \end{aligned}$$

Substituting for the $\dot{x}_2$ terms, we obtain the state-space representation

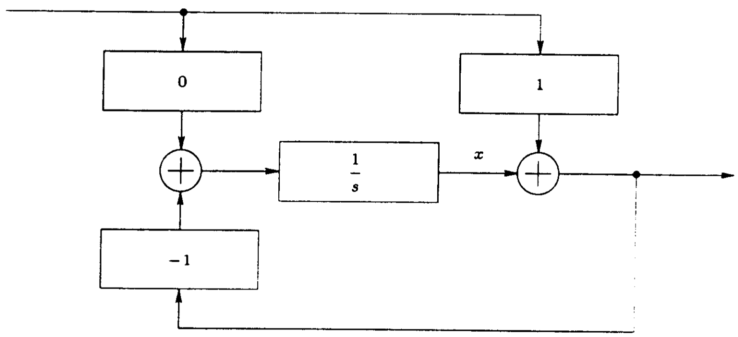
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/2 \end{bmatrix} u(t).$$

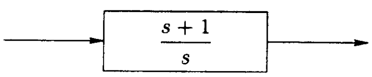
If we use the observer realization form for each of the blocks, then we obtain a different state-space representation.



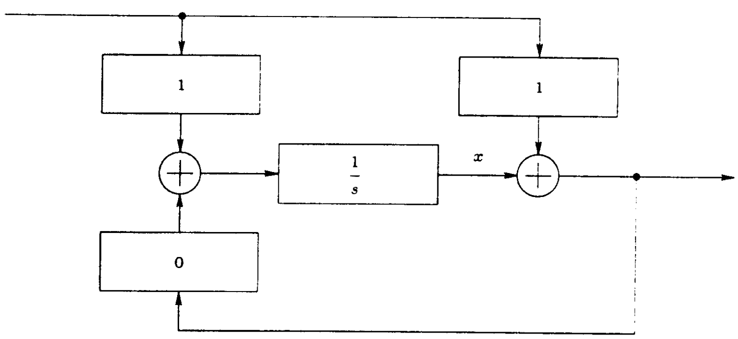
(a) The first feedforward gain block.



(b) Observer realization form.

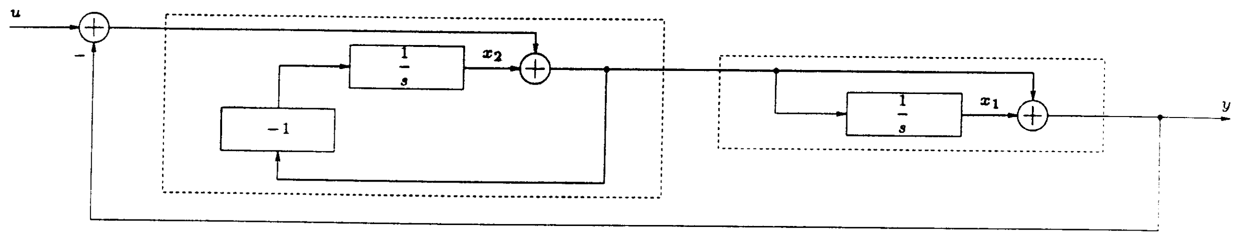


(a) The second feedforward gain block.



(b) Observer realization form.

The connected and "expanded" block diagram for this case is shown below.



Similarly, we obtain

$$\dot{x}_1 = x_2 + (u - y),$$
$$\dot{x}_2 = -\big(x_2 + (u - y)\big),$$

and

$$y = x_1 + \big(x_2 + (u - y)\big).$$

Solving for the output variable in the last equation, we get

$$\begin{aligned} y &= x_1 + x_2 + u - y \\ 2y &= x_1 + x_2 + u \\ y &= (1/2)x_1 + (1/2)x_2 + (1/2)u \end{aligned}$$

And,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/2 \end{bmatrix} u(t).$$

(b) Determine the transfer function of the system from its state-space representation.

Solution: The transfer function of a control system described in the state-state representation

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t),$$

is

$$F(s) = C(sI - A)^{-1}B + D;$$

where in our case

$$A = \begin{bmatrix} -1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1/2 & -1/2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1/2 \end{bmatrix},$$

in the controller form;

$$A = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1/2 \end{bmatrix},$$

in the observer form. Here, I is the appropriately dimensioned identity matrix.

Using the controller form,

$$F(s) = \begin{bmatrix} 1/2 & -1/2 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + 1/2$$

$$= \begin{bmatrix} 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} s+1/2 & 1/2 \\ 1/2 & s+1/2 \end{bmatrix}^{-1} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + 1/2$$

$$= \frac{1}{(s+1/2)^2 + 1/4} \begin{bmatrix} 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} s+1/2 & -1/2 \\ -1/2 & s+1/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + 1/2$$

$$= \frac{1}{(s+1/2)^2 + 1/4} \begin{bmatrix} 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1/2s \\ 1/2s \end{bmatrix} + 1/2$$

$$= \frac{1}{(s+1/2)^2 + 1/4} \begin{bmatrix} 0 \end{bmatrix} + 1/2.$$

In other words, the transfer function is $F(s) = 1/2$.

Using the observer form, we get the same result.

$$F(s) = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} + 1/2$$

$$= \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} s+1/2 & -1/2 \\ -1/2 & s+1/2 \end{bmatrix}^{-1} \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} + 1/2$$

$$= \frac{1}{(s+1/2)^2 + 1/4} \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} s+1/2 & 1/2 \\ 1/2 & s+1/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} + 1/2$$

$$= \frac{1}{(s+1/2)^2 + 1/4} \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2s \\ -1/2s \end{bmatrix} + 1/2$$

$$= \frac{1}{(s+1/2)^2 + 1/4} \begin{bmatrix} 0 \end{bmatrix} + 1/2 = 1/2.$$

3. A time-varying linear control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -6+5e^{-t} & 3-3e^{-t} \\ -10+10e^{-t} & 5-6e^{-t} \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

where u and $\boldsymbol{x}$ are the input and the state variables, respectively. Determine $\boldsymbol{x}(t)$ for $t \geq 0$, when $\boldsymbol{x}(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, and $u(t) = 0$ for $t \geq 0$.

Solution: Since the given system is time varying, the solution is given by

$$\boldsymbol{x}(t) = \Phi(t,0)\boldsymbol{x}(0) + \int_0^t \Phi(t,\tau)B(\tau)\boldsymbol{u}(\tau)\,\mathrm{d}\tau,$$

where Φ is the state-transition matrix, and B is the input matrix. When the state matrix A and its integral commute, or when the commutativity condition $A(t_1)A(t_2) = A(t_2)A(t_1)$ for all t_1 and t_2 is satisfied; the state-transition matrix is given by

$$\Phi(t,t_0) = e^{\left(\int_{t_0}^t A(\tau)\,\mathrm{d}\tau\right)}.$$

In our case,

$$A(t_1)A(t_2) = \begin{bmatrix} -6+5e^{-t_1} & 3-3e^{-t_1} \\ -10+10e^{-t_1} & 5-6e^{-t_1} \end{bmatrix} \begin{bmatrix} -6+5e^{-t_2} & 3-3e^{-t_2} \\ -10+10e^{-t_2} & 5-6e^{-t_2} \end{bmatrix}$$

$$= \begin{bmatrix} 6-5e^{-t_1-t_2} & -3+3e^{-t_1-t_2} \\ 10-10e^{-t_1-t_2} & -5+6e^{-t_1-t_2} \end{bmatrix} = A(t_2)A(t_1),$$

and the state-transition matrix is exponential of the integral of the state matrix. However, direct computation of the exponential is rather involved and a simpler approach is preferable.

Diagonalization of $A(t)$

One possible approach is to diagonalize the state matrix. If there exists a matrix T, such that

$$T^{-1}A(t)T = \Lambda(t)$$

is diagonal, then

$$\Phi(t,t_0) = e^{\left(\int_{t_0}^t A(\tau)\,\mathrm{d}\tau\right)} = Te^{\left(\int_{t_0}^t T^{-1}A(\tau)T\,\mathrm{d}\tau\right)}T^{-1} = Te^{\left(\int_{t_0}^t \Lambda_A(\tau)\,\mathrm{d}\tau\right)}T^{-1}.$$

To diagonalize the state matrix, we first need to find its eigenvalues from the characteristic equation

$$\det\big(\lambda I - A(t)\big) = 0,$$

or

$$\det \begin{bmatrix} \lambda+6-5e^{-t} & -3+3e^{-t} \\ 10-10e^{-t} & \lambda-5+6e^{-t} \end{bmatrix} = \lambda^2 + \left(1+e^{-t}\right)\lambda + e^{-t} = (\lambda+1)\left(\lambda+e^{-t}\right) = 0.$$

So, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -e^{-t}$.

For $\lambda_1 = -1$, we have

$$(-I - A)\boldsymbol{v}_1 = \begin{bmatrix} 5 - 5e^{-t} & -3 + 3e^{-t} \\ 10 - 10e^{-t} & -6 + 6e^{-t} \end{bmatrix} \begin{bmatrix} {v_1}^1 \\ {v_1}^2 \end{bmatrix} = 0,$$

or

$$(5 - 5e^{-t}){v_1}^1 + (-3 + 3e^{-t}){v_1}^2 = 0.$$

Letting ${v_1}^2 = 5$, we get $\boldsymbol{v}_1 = \begin{bmatrix} 3 & 5 \end{bmatrix}^T$.

Similarly, for $\lambda_2 = -e^{-t}$, we have

$$(-e^{-t}I - A)\boldsymbol{v}_2 = \begin{bmatrix} 6 - 6e^{-t} & -3 + 3e^{-t} \\ 10 - 10e^{-t} & -5 + 5e^{-t} \end{bmatrix} \begin{bmatrix} {v_2}^1 \\ {v_2}^2 \end{bmatrix} = 0,$$

or

$$(6 - 6e^{-t}){v_2}^1 + (-3 + 3e^{-t}){v_2}^2 = 0.$$

Letting ${v_2}^2 = 2$, we get $\boldsymbol{v}_2 = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$.

Therefore,

$$T = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

diagonalizes the state matrix, such that

$$T^{-1}A(t)T = \Lambda_A(t) = \begin{bmatrix} -1 & 0 \\ 0 & -e^{-t} \end{bmatrix}.$$

First taking the integral of Λ_A, we get

$$\int_{t_0}^{t} \Lambda_A(\tau)\,\mathrm{d}\tau = \begin{bmatrix} \int_{t_0}^{t}(-1)\,\mathrm{d}\tau & 0 \\ 0 & \int_{t_0}^{t}(-e^{-\tau})\,\mathrm{d}\tau \end{bmatrix} = \begin{bmatrix} -\tau & 0 \\ 0 & e^{-\tau} \end{bmatrix}_{\tau=t_0}^{\tau=t} = \begin{bmatrix} -(t - t_0) & 0 \\ 0 & e^{-t} - e^{-t_0} \end{bmatrix};$$

second taking the exponential of the integral, we get

$$e^{\left(\int_{t_0}^{t} \Lambda_A(\tau)\,\mathrm{d}\tau\right)} = \begin{bmatrix} e^{-(t-t_0)} & 0 \\ 0 & e^{(e^{-t} - e^{-t_0})} \end{bmatrix};$$

and finally,

$$\begin{aligned} \Phi(t,0) &= Te^{\left(\int_0^t \Lambda_A(\tau)\,\mathrm{d}\tau\right)}T^{-1} \\ &= \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{(e^{-t}-1)} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 6e^{-t} - 5e^{(e^{-t}-1)} & -3e^{-t} + 3e^{(e^{-t}-1)} \\ 10e^{-t} - 10e^{(e^{-t}-1)} & -5e^{-t} + 6e^{(e^{-t}-1)} \end{bmatrix}. \end{aligned}$$

Since $\boldsymbol{x}(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, and $u(t) = 0$ for $t \geq 0$; we have $\boldsymbol{x}(t) = \Phi(t,0)\boldsymbol{x}(0)$, or

$$\boldsymbol{x}(t) = \begin{bmatrix} 9e^{-t} - 8e^{(e^{-t}-1)} \\ 15e^{-t} - 16e^{(e^{-t}-1)} \end{bmatrix} \quad \text{for } t \geq 0.$$

Using the Cayley-Hamilton theorem

Another possible approach is to use the Cayley-Hamilton theorem to determine $\Phi(t, t_0)$ from the eigenvalues of $A(t)$. In this method, we observe that $\Phi(t, t_0)$ may be described by a linear combination of $A^k(t)$ for $k = 0, \ldots, (n-1)$, so that

$$e^{\left(\int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau\right)} = \alpha_0(t)I + \alpha_1(t)A(t) + \ldots + \alpha_{n-1}(t)A^{n-1}(t),$$

where I is the appropriately dimensioned identity matrix, n is the dimension of the system. and $\alpha_0(t), \ldots, \alpha_{n-1}(t)$ are functions of time. The scalars are determined by the application of the eigenvectors to the above equation that results in the set of equations

$$e^{\left(\int_0^t \lambda_1(\tau)\,\mathrm{d}\tau\right)} = \alpha_0(t)I + \alpha_1(t)\lambda_1(t) + \ldots + \alpha_{n-1}(t)\lambda_1^{n-1}(t)$$

$$\vdots$$

$$e^{\left(\int_0^t \lambda_n(\tau)\,\mathrm{d}\tau\right)} = \alpha_0(t)I + \alpha_1(t)\lambda_n(t) + \ldots + \alpha_{n-1}(t)\lambda_n^{n-1}(t),$$

where $\lambda_1(t), \ldots, \lambda_n(t)$ are the eigenvalues. Since we have already determined the eigenvalues of $A(t)$ as $\lambda_1(t) = -1$ and $\lambda_2(t) = -e^{-t}$, and $n = 2$; the set of equations becomes

$$e^{\left(\int_0^t (-1)\,\mathrm{d}\tau\right)} = \alpha_0(t) + \alpha_1(t)(-1)$$
$$e^{\left(\int_0^t (-e^{-\tau})\,\mathrm{d}\tau\right)} = \alpha_0(t) + \alpha_1(t)(-e^{-t}),$$

or

$$e^{-t} = \alpha_0(t) - \alpha_1(t)$$
$$e^{(e^{-t}-1)} = \alpha_0(t) - e^{-t}\alpha_1(t).$$

Solving the above set of equations simultaneously gives

$$\alpha_0(t) = \left(e^{-2t} - e^{(e^{-t}-1)}\right) \Big/ \left(e^{-t} - 1\right)$$
$$\alpha_1(t) = \left(e^{-t} - e^{(e^{-t}-1)}\right) \Big/ \left(e^{-t} - 1\right)$$

As a result,

$$\Phi(t, 0) = \alpha_0(t)I + \alpha_1(t)A(t),$$

and

$$\begin{aligned}
\boldsymbol{x}(t) &= \Phi(t,0)\boldsymbol{x}(0) = \alpha_0(t)\boldsymbol{x}(0) + \alpha_1(t)A(t)\boldsymbol{x}(0) \\
&= \left(\frac{e^{-2t} - e^{(e^{-t}-1)}}{e^{-t}-1}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \left(\frac{e^{-t} - e^{(e^{-t}-1)}}{e^{-t}-1}\right)\begin{bmatrix} -9 + 8e^{-t} \\ -15 + 16e^{-t} \end{bmatrix} \\
&= \left(\frac{1}{e^{-t}-1}\right)\begin{bmatrix} 9e^{-2t} - 9e^{-t} - 8e^{(e^{-t}-1)}e^{-t} + 8e^{(e^{-t}-1)} \\ 15e^{-2t} - 15e^{-t} - 16e^{(e^{-t}-1)}e^{-t} + 16e^{(e^{-t}-1)} \end{bmatrix} \\
&= \left(\frac{1}{e^{-t}-1}\right)\begin{bmatrix} 9e^{-t}(e^{-t}-1) - 8e^{(e^{-t}-1)}(e^{-t}-1) \\ 15e^{-t}(e^{-t}-1) - 16e^{(e^{-t}-1)}(e^{-t}-1) \end{bmatrix} \\
&= \begin{bmatrix} 9e^{-t} - 8e^{(e^{-t}-1)} \\ 15e^{-t} - 16e^{(e^{-t}-1)} \end{bmatrix},
\end{aligned}$$

which is the same result as before.

4. A control system is described in state-space representation, such that

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t),$$

where $\boldsymbol{u}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ are the input, the state, and the output variables, respectively. For the following A, B, C, and D matrices, determine whether the system is asymptotically stable, marginally stable, or unstable; and whether it is bounded-input-bounded-output stable or not. Justify your answer.

(a)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and } D = 0.$$

Solution: In order to determine the stability of the system, we first need to determine its eigenvalues or poles. Since in this case, the state matrix A is diagonal, we observe the eigenvalues directly from the diagonal elements as $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = -1$. The eigenvalue λ_3 has a negative real part, and it would generate an asymptotically stable response. The eigenvalues λ_1 and λ_2 are both zero, and each would generate a constant response individually. However, if they are cascaded, a constant response generated by the first one would result in a ramp response by the second one. In this case, since the state matrix A is diagonal, the two zero-valued eigenvalues don't affect each other, or they are not cascaded. Therefore, each of the eigenvalues λ_1 and λ_2 would generate a constant response resulting in a marginally stable response. Since there are no more eigenvalues, we conclude that the system is marginally stable.

In order to determine the bounded-input-bounded-output stability of the system, we may determine the transfer matrix and observe the poles of the system after all the reductions. The transfer matrix of the system is given by

$$\mathcal{L}\left[\boldsymbol{y}\right](s) = \left(C(sI - A)^{-1}B + D\right)\mathcal{L}\left[\boldsymbol{u}\right](s)$$

where $\mathcal{L}\left[(\cdot)\right](s)$ is the Laplace transform, and I is the appropriately dimensioned identity matrix. In our case,

$$C(sI - A)^{-1}B + D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/s & 0 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1/(s+1) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/s \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/s \end{bmatrix}.$$

We realize that the only pole that is visible in the transfer matrix is on the imaginary axis. Since that pole would generate a ramp response for a step input, the system is not bounded-input-bounded-output stable.

In summary, the system is marginally stable, and it is not bounded-input-bounded-output stable.

(b)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \text{ and } D = 0.$$

Solution: Since the state matrix A is upper diagonal, we observe the eigenvalues directly from the diagonal elements as $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = -1$. The eigenvalue λ_3 has a negative real part, and it would generate an asymptotically stable response. The eigenvalues λ_1 and λ_2 are both zero, and each would generate a constant response individually. However, if they are cascaded, a constant response generated by the first one would result in a ramp response by the second one. In this case, since the state matrix A is in Jordan form, the two zero-valued eigenvalues are cascaded, and they affect each other. Therefore, the state corresponding to λ_2 would generate a constant response, and the state corresponding to λ_1 would then generate a ramp response. As a result, we conclude that the system is unstable.

In order to determine the bounded-input-bounded-output stability of the system, we may determine the transfer matrix and observe the poles of the system after all the reductions. The transfer matrix of the system is given by

$$\mathcal{L}\left[\boldsymbol{y}\right](s) = \left(C(sI - A)^{-1}B + D\right)\mathcal{L}\left[\boldsymbol{u}\right](s)$$

where $\mathcal{L}\left[(\cdot)\right](s)$ is the Laplace transform, and I is the appropriately dimensioned identity matrix. One method to determine the inverse of $(sI - A)$ is to use row operations on the augmented matrix $\left[\,(sI - A) \quad I\,\right]$ to generate $\left[\,I \quad (sI - A)^{-1}\,\right]$.

$$\left[\begin{array}{ccc|ccc} s & -1 & 0 & 1 & 0 & 0 \\ 0 & s & 0 & 0 & 1 & 0 \\ 0 & 0 & s+1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} s & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/s & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/(s+1) \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} s & 0 & 0 & 1 & 1/s & 0 \\ 0 & 1 & 0 & 0 & 1/s & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/(s+1) \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/s & 1/s^2 & 0 \\ 0 & 1 & 0 & 0 & 1/s & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/(s+1) \end{array}\right].$$

Therefore,

$$C(sI - A)^{-1}B + D = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/s & 1/s^2 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1/(s+1) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/s \\ 0 \\ 1/(s+1) \end{bmatrix} = \begin{bmatrix} 1/s + 1/(s+1) \\ 1/s \end{bmatrix}.$$

Because of the pole at zero, the step response would contain a ramp function; and as a result the system is not bounded-input-bounded-output stable.

In summary, the system is unstable, and it is not bounded-input-bounded-output stable.