EE 431 Exam#1 Oct. 22, 2009

Solutions

1. Determine

$$A = \cos\begin{bmatrix} \lambda & 0 & 0 & 1 \\ 0 & \lambda & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

Show your work and simplify the expressions as much as possible.

Solution: We can compute a matrix function, that has a non-trivial Taylor's Series Expansion, by its expansion along with the use of the Cayley-Hamilton's Theorem or by the use of a simplifying transformation, where

$$f(B) = Qf(Q^{-1}BQ)Q^{-1}$$

for a transformation Q, such that $Q^{-1}BQ$ is diagonal or in jordan form, and the evaluation of $f(Q^{-1}BQ)$ is directly performed.

In this case, the matrix is almost in block-diagonal jordan form, since the first and the fourth eigenvalues are related, as well as the third and the second eigenvalues.

We let $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ be a vector in the original space and $\tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 \end{bmatrix}^T$ be the transformed vector. To bring the related first and the fourth eigenvalues next to each other, we let

$$\tilde{x}_1 = x_1,$$
$$\tilde{x}_2 = x_4.$$

Then we place the third and the second eigenvalues next by letting

$$\tilde{x}_3 = x_3,$$
$$\tilde{x}_4 = x_2.$$

Here, we swapped the order of the original placement, since we want the identity element below the diagonal to be above the diagonal in the jordan form. Rewriting the above equations in matrix form, we have

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix};$$

or by taking the inverse of the coefficient matrix, we obtain the transformation matrix

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} = Q\tilde{\boldsymbol{x}}.$$

Therefore,

$$A = Q\Big(\cos\big(Q^{-1}BQ\big)\Big)Q^{-1},$$

where

$$Q^{-1}BQ = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

As a result, we get

$$A = \cos\begin{bmatrix} \lambda & 0 & 0 & 1 \\ 0 & \lambda & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} = Q\left(\cos\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}\right)Q^{-1}$$

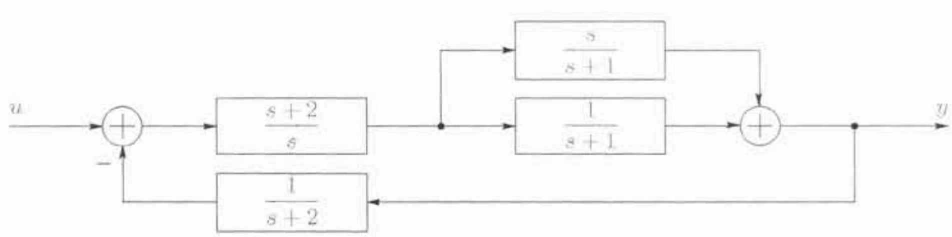
$$= Q\begin{bmatrix} \cos(\lambda) & \mathrm{d}\big(\cos(\lambda)\big)/\mathrm{d}\lambda & 0 & 0 \\ 0 & \cos(\lambda) & 0 & 0 \\ 0 & 0 & \cos(\lambda) & \mathrm{d}\big(\cos(\lambda)\big)/\mathrm{d}\lambda \\ 0 & 0 & 0 & \cos(\lambda) \end{bmatrix}Q^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos(\lambda) & -\sin(\lambda) & 0 & 0 \\ 0 & \cos(\lambda) & 0 & 0 \\ 0 & 0 & \cos(\lambda) & -\sin(\lambda) \\ 0 & 0 & 0 & \cos(\lambda) \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1}.$$

After simplifying the above expression, we get

$$A = \begin{bmatrix} \cos(\lambda) & 0 & 0 & -\sin(\lambda) \\ 0 & \cos(\lambda) & 0 & 0 \\ 0 & -\sin(\lambda) & \cos(\lambda) & 0 \\ 0 & 0 & 0 & \cos(\lambda) \end{bmatrix}.$$

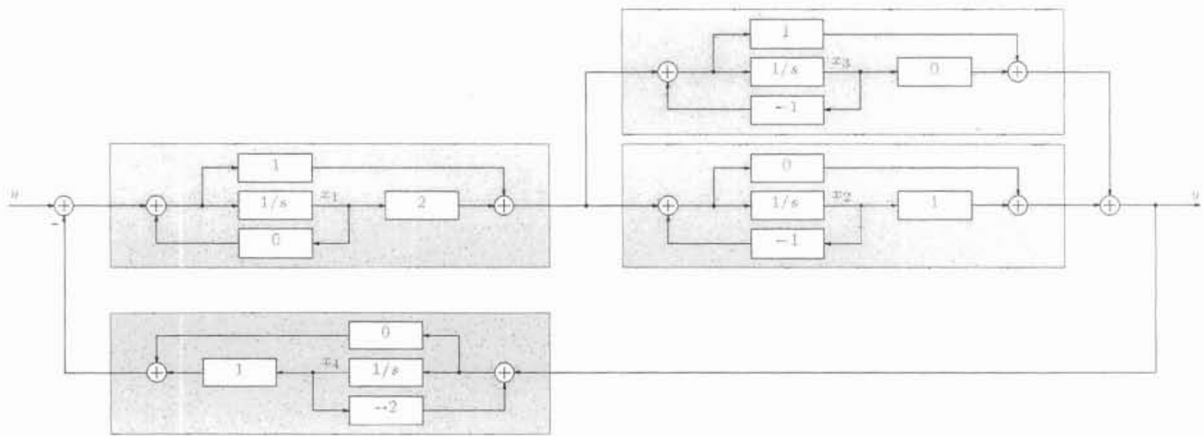
2. The block diagram of a control system is given below.



Obtain a state-space representation of the system without any block-diagram reduction.

Solution: In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.

We may use a number of possible forms to realize the blocks. If we use the controller canonical form, we get the following diagram.



After assigning the state variables as shown in the figure, we obtain

$$\begin{aligned}
\dot{x}_1 &= (0)x_1 + \Big(u - \big((1)x_4 + (0)\dot{x}_4\big)\Big) = -x_4 + u,\\
\dot{x}_2 &= (-1)x_2 + \Big((2)x_1 + (1)\dot{x}_1\Big) = 2x_1 - x_2 - x_4 + u,\\
\dot{x}_3 &= (-1)x_3 + \Big((2)x_1 + (1)\dot{x}_1\Big) = 2x_1 - x_3 - x_4 + u,\\
\dot{x}_4 &= (-2)x_4 + y,
\end{aligned}$$

and

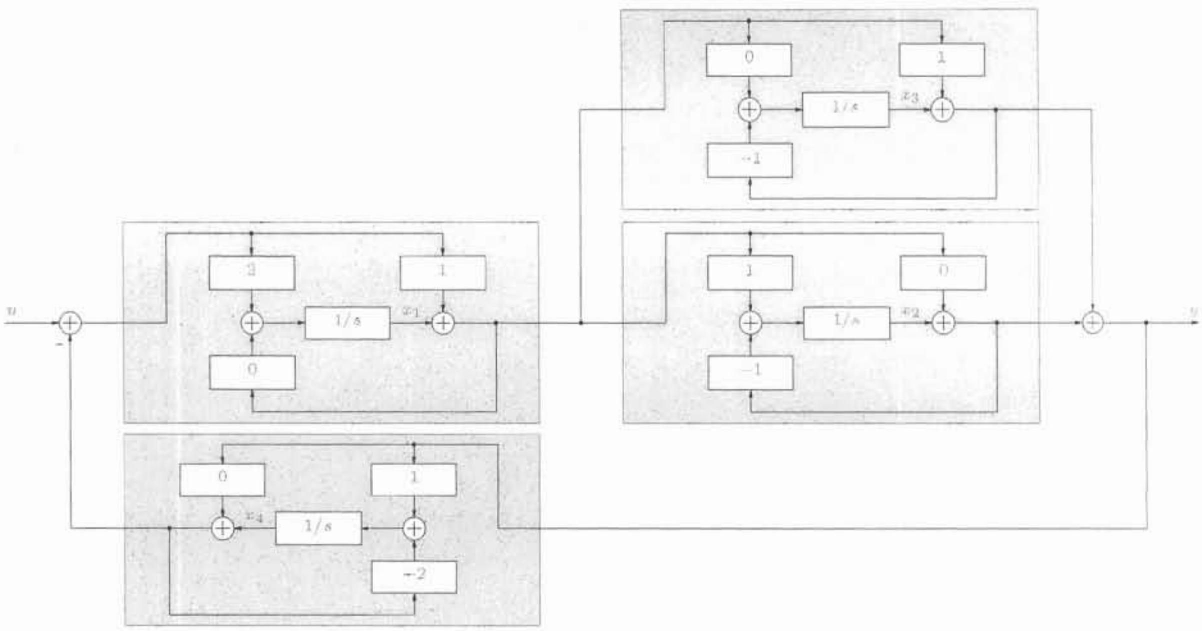
$$y = \Big((1)x_2 + (0)\dot{x}_2\Big) + \Big((0)x_3 + (1)\dot{x}_3\Big) = x_2 + \dot{x}_3 = 2x_1 + x_2 - x_3 - x_4 + u.$$

After substituting for y in the $\dot{x}_4$ equation, we obtain the state-space representation based on the controller canonical form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 2 & -1 & 0 & -1 \\ 2 & 0 & -1 & -1 \\ 2 & 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 2 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} u(t).$$

If we use the observer realization form for each of the blocks, then we obtain a different state-space representation.



Similarly, we obtain

$$\begin{aligned}
\dot{x}_1 &= (0)y_1 + (2)\Big(u - \big(x_4 + (0)y\big)\Big) = -2x_4 + 2u, \\
\dot{x}_2 &= (-1)\big(x_2 + (0)y_1\big) + (1)y_1 = -x_2 + y_1, \\
\dot{x}_3 &= (-1)\big(x_3 + (1)y_1\big) + (0)y_1 = -x_3 - y_1, \\
\dot{x}_4 &= (-2)\big(x_4 + (0)y\big) + (1)y = -2x_4 + y,
\end{aligned}$$

and

$$y = \big(x_2 + (0)y_1\big) + \big(x_3 + (1)y_1\big) = x_2 + x_3 + y_1,$$

where

$$y_1 = x_1 + (1)\Big(u - \big(x_4 + (0)y\big)\Big) = x_1 - x_4 + u.$$

After substituting y_1 and y into the equations, we get the state-space representation based on the observer canonical form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 1 & -1 & 0 & -1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} u(t).$$

3. A control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -4 & 1 & -1 \\ -1 & -2 & -1 \\ -4 & -4 & -10 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \boldsymbol{x}(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively.

(a) Determine $\boldsymbol{x}(t)$ for $t \geq 0$, when $\boldsymbol{x}(0) = \begin{bmatrix} 0 & 0 & 9 \end{bmatrix}^T$, and $u(t) = 0$ for $t \geq 0$.

Solution: The general solution to the state-space representation of a system described by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t)$$

is obtained from

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)} B\boldsymbol{u}(\tau)\, \mathrm{d}\tau,$$

where I is the appropriately dimensioned identity matrix.

In our case,

$$A = \begin{bmatrix} -4 & 1 & -1 \\ -1 & -2 & -1 \\ -4 & -4 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix},$$

and $u(t) = 0$ for $t \geq 0$. As a result, the integral term in the solution of $\boldsymbol{x}$ is identically zero. Moreover, since the initial condition $\boldsymbol{x}(0)$ has the first two elements zero, we need to calculate only the third row of the state transition matrix e^{At}.

We may use a lot of different approaches to determine the state transition matrix, such as the Taylor's Series Expansion along with the Cayley-Hamilton Theorem, or a simplifying transformation, or the inverse Laplace Transform of $(sI - A)^{-1}$. However, since we need to only compute the third row, the inverse Laplace Transform approach would be the easiest.

$$e^{At} = \mathcal{L}_s^{-1}\left[\,(sI-A)^{-1}\,\right](t)$$

$$= \mathcal{L}_s^{-1}\left[\left(s\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}-4&1&-1\\-1&-2&-1\\-4&-4&-10\end{bmatrix}\right)^{-1}\right](t)$$

$$= \mathcal{L}_s^{-1}\left[\begin{bmatrix}s+4&-1&1\\1&s+2&1\\4&4&s+10\end{bmatrix}^{-1}\right](t)$$

$$= \mathcal{L}_s^{-1}\left[\frac{1}{(s+2)(s+3)(s+11)}\begin{bmatrix}s^2+12s+16&s+14&-(s+3)\\-(s+6)&s^2+14s+36&-(s+3)\\-4(s+1)&-4(s+5)&(s+3)^2\end{bmatrix}\right](t)$$

$$= \begin{bmatrix}*&*&\mathcal{L}_s^{-1}\left[\frac{-1}{(s+2)(s+11)}\right](t)\\ *&*&\mathcal{L}_s^{-1}\left[\frac{-1}{(s+2)(s+11)}\right](t)\\ *&*&\mathcal{L}_s^{-1}\left[\frac{s+3}{(s+2)(s+11)}\right](t)\end{bmatrix}$$

$$= \begin{bmatrix}*&*&\mathcal{L}_s^{-1}\left[\frac{-1/9}{s+2}+\frac{1/9}{s+11}\right](t)\\ *&*&\mathcal{L}_s^{-1}\left[\frac{-1/9}{s+2}+\frac{1/9}{s+11}\right](t)\\ *&*&\mathcal{L}_s^{-1}\left[\frac{1/9}{s+2}+\frac{8/9}{s+11}\right](t)\end{bmatrix}$$

$$= \begin{bmatrix}*&*&-(1/9)e^{-2t}+(1/9)e^{-11t}\\ *&*&-(1/9)e^{-2t}+(1/9)e^{-11t}\\ *&*&(1/9)e^{-2t}+(8/9)e^{-11t}\end{bmatrix}.$$

As a result,

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) = \begin{bmatrix}*&*&-(1/9)e^{-2t}+(1/9)e^{-11t}\\ *&*&-(1/9)e^{-2t}+(1/9)e^{-11t}\\ *&*&(1/9)e^{-2t}+(8/9)e^{-11t}\end{bmatrix}\begin{bmatrix}0\\0\\9\end{bmatrix},$$

or

$$\boldsymbol{x}(t) = \begin{bmatrix}-e^{-2t}+e^{-11t}\\-e^{-2t}+e^{-11t}\\e^{-2t}+8e^{-11t}\end{bmatrix} \text{ for } t \geq 0.$$

(b) Determine the transfer function of the system. Show your work clearly.

Solution: The transfer function of a control system described in the state-state representation

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t),$$

is

$$H(s) = C(sI - A)^{-1}B + D;$$

where in our case

$$A = \begin{bmatrix} -4 & 1 & -1 \\ -1 & -2 & -1 \\ -4 & -4 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}.$$

Here, I is the appropriately dimensioned identity matrix.

Therefore,

$$H(s) = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} s+4 & -1 & 1 \\ 1 & s+2 & 1 \\ 4 & 4 & s+10 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \left(\frac{1}{(s+2)(s+3)(s+11)}\right)$$
$$\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} s^2+12s+16 & s+14 & -(s+3) \\ -(s+6) & s^2+14s+36 & -(s+3) \\ -4(s+1) & -4(s+5) & (s+3)^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \left(\frac{1}{(s+2)(s+3)(s+11)}\right) \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} s^2+13s+30 \\ s^2+13s+30 \\ -8s-24 \end{bmatrix}$$
$$= \frac{1}{(s+2)(s+3)(s+11)} \begin{bmatrix} 0 \end{bmatrix}.$$

In other words, the transfer function is $H(s) = 0$.

(c) Determine the Markov Parameters of the system. Show your work clearly.

Solution: The markov parameters of a control system described in the state-state representation

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t),$$

is given by

$$h_0 = D \text{ and } h_i = CA^{i-1}B$$

for $i = 1, \ldots$, where in our case

$$A = \begin{bmatrix} -4 & 1 & -1 \\ -1 & -2 & -1 \\ -4 & -4 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}.$$

Here, I is the appropriately dimensioned identity matrix.

By direct substitution, we get

$$h_0 = 0,$$

$$\begin{aligned} h_1 &= CB = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0, \\ h_2 &= CAB = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -4 & 1 & -1 \\ -1 & -2 & -1 \\ -4 & -4 & -10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0, \\ h_3 &= CA^2B = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -4 & 1 & -1 \\ -1 & -2 & -1 \\ -4 & -4 & -10 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0, \end{aligned}$$

and

$$h_4 = h_5 = h_6 = \ldots = 0$$

due to the Cayley-Hamilton's Theorem. In other words, the markov paramaters are all zero.

Indeed, this result is also obvious from the transfer function, since

$$\begin{aligned} H(s) &= D + C(sI - A)^{-1}B \\ &= h_0 + h_1 \frac{1}{s} + h_2 \frac{1}{s^2} + h_3 \frac{1}{s^3} + \cdots \\ &= 0. \end{aligned}$$

4. A time-varying linear control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} a\sin(t) & t \\ 0 & a\sin(t) \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

where u and $\boldsymbol{x}$ are the input and the state variables, respectively. Determine the state-transition matrix $\Phi(t, t_0)$.

Solution: In this case, the state matrix A is upper triangular, so both eigenvalues are at $\lambda_{1,2}(t) = a\sin(t)$.

When the state matrix and its integral commute, or when the commutativity condition $A(t_1)A(t_2) = A(t_2)A(t_1)$ is satisfied for all t_1 and t_2; the state-transition matrix is given by

$$\Phi(t,t_0) = e^{\left(\int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau\right)}.$$

In our case the commutativity condition holds, since.

$$\begin{aligned} A(t_1)A(t_2) &= \begin{bmatrix} a\sin(t_1) & t_1 \\ 0 & a\sin(t_1) \end{bmatrix}\begin{bmatrix} a\sin(t_2) & t_2 \\ 0 & a\sin(t_2) \end{bmatrix} \\ &= \begin{bmatrix} a^2\sin(t_1)\sin(t_2) & at_2\sin(t_1)+at_1\sin(t_2) \\ 0 & a^2\sin(t_1)\sin(t_2) \end{bmatrix} = A(t_2)A(t_1). \end{aligned}$$

We may use the Cayley-Hamilton's Theorem to determine the state-transition matrix directly, such that

$$\Phi(t,t_0) = e^{\left(\int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau\right)} = \alpha_0 I + \alpha_1 A(t)$$

for some α_0 and α_1.

However, there's a difficulty with the application of the theorem, when there are repeated eigenvalues in the time-varying case. The set of equations for the repeated eigenvalues are

$$\left[e^{\left(\int_{t_0}^{t} \lambda(\tau)\,\mathrm{d}\tau\right)} \right]_{\lambda(\tau)=a\sin(\tau)} = \Big[\alpha_0 + \alpha_1\lambda(t) \Big]_{\lambda(t)=a\sin(t)},$$

and

$$\left[\frac{\mathrm{d}\left(e^{\left(\int_{t_0}^{t} \lambda(\tau)\,\mathrm{d}\tau\right)}\right)}{\mathrm{d}\lambda(t)} \right]_{\lambda(\tau)=a\sin(\tau)} = \left[\frac{\mathrm{d}\big(\alpha_0+\alpha_1\lambda(t)\big)}{\mathrm{d}\lambda(t)} \right]_{\lambda(t)=a\sin(t)}.$$

As we may observe, there's a problem with the derivative of the exponential term, since the exponential term has the variable $\lambda(\tau)$, whereas the derivative is with respect to $\lambda(t)$.

There's no easy way to fix this difficulty as long as the time-varying eigenvalue or the time-varying state matrix is inside the integral.

In order to consider a case where the time-varying state matrix is not inside an integral, we may work on the integral of the matrix instead of the matrix itself. In other words, we may let

$$\begin{aligned} A_{\text{int}}(t,t_0) &= \int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau = \int_{t_0}^{t} \begin{bmatrix} a\sin(\tau) & \tau \\ 0 & a\sin(\tau) \end{bmatrix} \mathrm{d}\tau \\ &= \begin{bmatrix} -a\cos(\tau) & \tau^2/2 \\ 0 & -a\cos(\tau) \end{bmatrix}_{\tau=t_0}^{\tau=t} \\ &= \begin{bmatrix} -a\big(\cos(t)-\cos(t_0)\big) & t^2/2 - {t_0}^2/2 \\ 0 & -a\big(\cos(t)-\cos(t_0)\big) \end{bmatrix} \end{aligned}$$

The eigenvalues of A_{int} are both at $\lambda_{\text{int}1,2}(t,t_0) = -a\big(\cos(t)-\cos(t_0)\big)$.

Now, the Cayley-Hamilton's Theorem may be used to determine the state-transition matrix from A_{int}, such that

$$\Phi(t,t_0) = e^{\left(\int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau\right)} = e^{\left(A_{\text{int}}(t,t_0)\right)} = \alpha_{\text{int}0} I + \alpha_{\text{int}1} A_{\text{int}}(t,t_0)$$

for some $\alpha_{\text{int}0}$ and $\alpha_{\text{int}1}$.

Application of the eigenvectors gives the two equations to solve for the unknown scalars, such that

$$e^{\left(\lambda_{\text{int}}(t,t_0)\right)} = \alpha_{\text{int}0} + \alpha_{\text{int}1}\lambda_{\text{int}}(t,t_0),$$

and

$$\frac{\mathrm{d}\left(e^{\left(\lambda_{\text{int}}(t,t_0)\right)}\right)}{\mathrm{d}\lambda_{\text{int}}(t,t_0)} = \frac{\mathrm{d}\left(\alpha_{\text{int}0} + \alpha_{\text{int}1}\lambda_{\text{int}}(t,t_0)\right)}{\mathrm{d}\lambda_{\text{int}}(t,t_0)},$$

$$e^{\left(\lambda_{\text{int}}(t,t_0)\right)} = \alpha_{\text{int}1}.$$

In our case, we get

$$\left[\, e^{\left(\lambda_{\text{int}}(t,t_0)\right)} = \alpha_{\text{int}0} + \alpha_{\text{int}1}\lambda_{\text{int}}(t,t_0) \,\right]_{\left\{\lambda_{\text{int}}(t,t_0) = -a\left(\cos(t)-\cos(t_0)\right)\right\}},$$

$$\left[\, e^{\left(\lambda_{\text{int}}(t,t_0)\right)} = \alpha_{\text{int}1} \,\right]_{\left\{\lambda_{\text{int}}(t,t_0) = -a\left(\cos(t)-\cos(t_0)\right)\right\}};$$

or

$$e^{-a\left(\cos(t)-\cos(t_0)\right)} = \alpha_{\text{int}0} - \alpha_{\text{int}1} a\big(\cos(t)-\cos(t_0)\big),$$

$$e^{-a\left(\cos(t)-\cos(t_0)\right)} = \alpha_{\text{int}1}.$$

Solving for the first variable, we get

$$\alpha_{\text{int}0} = \Big(1 + a\big(\cos(t)-\cos(t_0)\big)\Big) e^{-a\left(\cos(t)-\cos(t_0)\right)},$$

$$\alpha_{\text{int}1} = e^{-a\left(\cos(t)-\cos(t_0)\right)}.$$

As a result, the state-transition matrix is

$$\begin{aligned}\Phi(t,t_0) &= e^{\left(A_{\text{int}}(t,t_0)\right)} = \alpha_{\text{int}0} I + \alpha_{\text{int}1} A_{\text{int}}(t,t_0)\\ &= \Big(1 + a\big(\cos(t)-\cos(t_0)\big)\Big) e^{-a\left(\cos(t)-\cos(t_0)\right)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\\ &\quad + e^{-a\left(\cos(t)-\cos(t_0)\right)} \begin{bmatrix} -a\big(\cos(t)-\cos(t_0)\big) & t^2/2 - {t_0}^2/2 \\ 0 & -a\big(\cos(t)-\cos(t_0)\big) \end{bmatrix},\end{aligned}$$

or

$$\Phi(t,t_0) = e^{-a\left(\cos(t)-\cos(t_0)\right)} \begin{bmatrix} 1 & t^2/2 - {t_0}^2/2 \\ 0 & 1 \end{bmatrix}.$$