**EE 431** **Exam#1** **Oct. 22, 2002**

75 minutes



1. Consider a space $P_2$ of polynomials with the basis $\{1, x, x^2\}$ and a mapping $\mathcal{P} : P_2 \to P_2$, such that

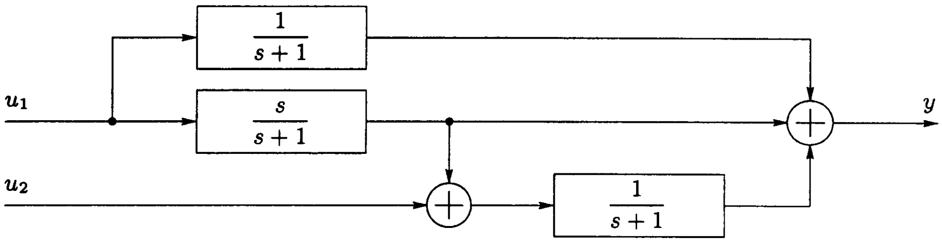
$$\mathcal{P}(f) = 2f' + 3f$$

for $f \in P_2$, where $(\cdot)'$ denotes the derivative.

   (a) Determine the matrix representation of the linear mapping $\mathcal{P}$ with respect to the given basis. (10pts)

   (b) Determine the null space of the mapping in terms of the given basis. (05pts)

2. The block diagram of a control system is given below.




Obtain a state-space representation of the system without any block-diagram reduction. (25pts)

3. A control system is described in state-space representation, such that

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & -4 & 0 \\ 0 & -2 & 0 \\ -2 & -1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} u(t),$$

where $u$ and $\mathbf{x}$ are the input and the state vectors, respectively.

   (a) Determine the poles of the system. (05pts)

   (b) Obtain the solution $x(t)$ for $t \geq 0$; when $x(0) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$, and $u(t) = 1$ for $t \geq 0$. (20pts)

4. A control system is described in state-space representation, such that

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} -2 & 1 \end{bmatrix} \mathbf{x}(t),$$

where $u$, $\mathbf{x}$, and $y$ are the input, the state, and the output variables, respectively. Determine the transfer function or the transfer matrix of the system. (15pts)

5. A time-varying control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \sin(t) & \cos(t) \\ 0 & \sin(t) \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{x}$ is the state variable. Determine $\mathbf{x}(2\pi)$, when $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. (20pts)

1. Consider a space $P_2$ of polynomials with the basis $\{1, x, x^2\}$ and a mapping $\mathcal{P} : P_2 \to P_2$, such that

$$\mathcal{P}(f) = 2f' + 3f$$

for $f \in P_2$, where $(\cdot)'$ denotes the derivative.

(a) Determine the matrix representation of the linear mapping $\mathcal{P}$ with respect to the given basis.

**Solution:** For any function $f \in P_2$, we can express $f$ such that $f(x) = f_0(1) + f_1(x) + f_2(x^2)$ for some constants $f_0$, $f_1$, and $f_2$. In vector notation, the vector $\mathbf{f} = \begin{bmatrix} f_0 & f_1 & f_2 \end{bmatrix}^T$ represents the function $f$. As a result

$$\begin{aligned}\mathcal{P}(f(x)) &= 2\big(\mathrm{d}f(x)/\mathrm{d}x\big) + 3f(x) \\ &= 2\big(f_1 + 2f_2x\big) + 3\big(f_0 + f_1x + f_2x^2\big) \\ &= (3f_0 + 2f_1)(1) + (3f_1 + 4f_2)(x) + (3f_2)(x^2).\end{aligned}$$

In vector notation, $\mathcal{P}(f)$ will be represented by

$$P\mathbf{f} = \begin{bmatrix} 3f_0 + 2f_1 \\ 3f_1 + 4f_2 \\ 3f_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}.$$

Therefore, the matrix representation of the mapping $\mathcal{P}$ with respect to the given basis is

$$P = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

Indeed, a more direct way of obtaining the matrix $P$ would be to observe that the columns of the matrix $P$ are formed from the mappings of the individual basis elements. The first basis element is 1, so

$$\begin{aligned}\mathcal{P}(1) &= 2\big(\mathrm{d}(1)/\mathrm{d}x\big) + 3(1) = 3 \\ &= 3(1) + 0(x) + 0(x^2),\end{aligned}$$

and the first column is $\begin{bmatrix} 3 & 0 & 0 \end{bmatrix}^T$. The second basis element is $x$, so

$$\begin{aligned}\mathcal{P}(x) &= 2\big(\mathrm{d}(x)/\mathrm{d}x\big) + 3(x) = 2 + 3x \\ &= 2(1) + 3(x) + 0(x^2),\end{aligned}$$

and the second column is $\begin{bmatrix} 2 & 3 & 0 \end{bmatrix}^T$. Finally, the third basis element is $x^2$, so

$$\begin{aligned}\mathcal{P}(x^2) &= 2\big(\mathrm{d}(x^2)/\mathrm{d}x\big) + 3(x^2) = 4x + 3x^2 \\ &= 0(1) + 4(x) + 3(x^2),\end{aligned}$$

and the third column is $\begin{bmatrix} 0 & 4 & 3 \end{bmatrix}^T$.

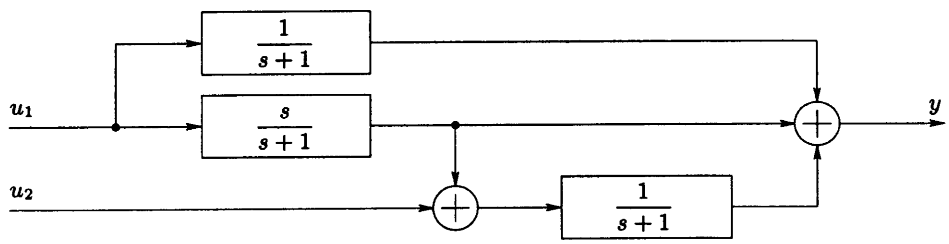
(b) Determine the null space of the mapping in terms of the given basis.

**Solution:** The mapping is represented by the matrix

$$P = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix},$$

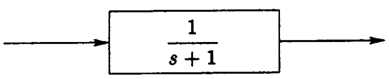
which is upper triangular with eigenvalues 3, 3, and 3. Since all the eigenvalues of $P$ are non-zero, the null space of the mapping is $\{\mathbf{0}\}$.

2. The block diagram of a control system is given below.

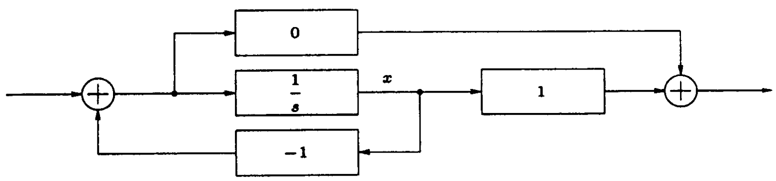



Obtain a state-space representation of the system without any block-diagram reduction.

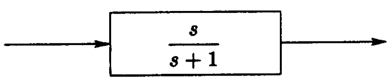
**Solution:** In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.



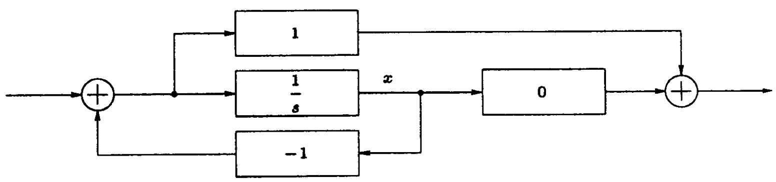

(a) The first gain block.




(b) Controller realization form.

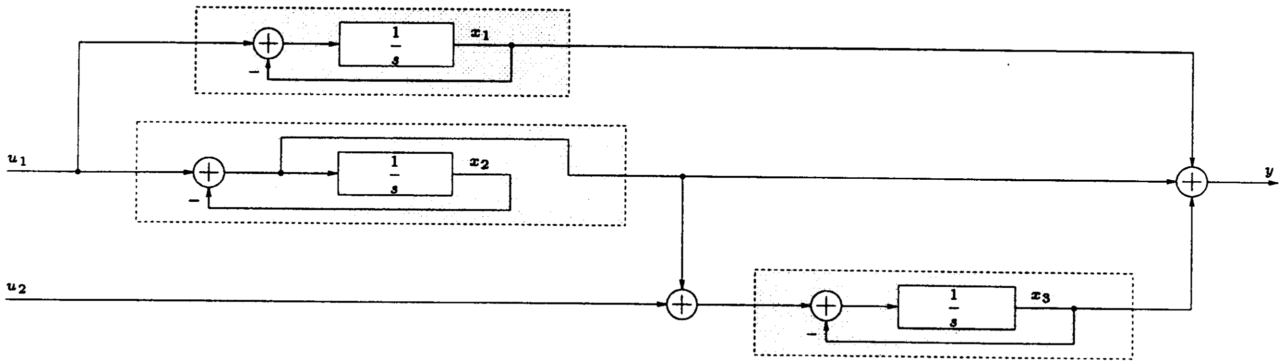



(c) The second gain block.




(d) Controller realization form.

The connected and "expanded" block diagram is shown below.




After assigning the state variables as shown in the figure, we obtain

$$\dot{x}_1 = -x_1 + u_1,$$

$$\dot{x}_2 = -x_2 + u_1,$$

$$\dot{x}_3 = -x_3 + (u_2 + \dot{x}_2) = -x_2 - x_3 + u_1 + u_2,$$

and

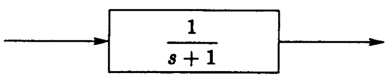
$$y = x_1 + \dot{x}_2 + x_3 = x_1 - x_2 + x_3 + u_1.$$

After expressing the above equations in matrix form, we obtain the state-space representation

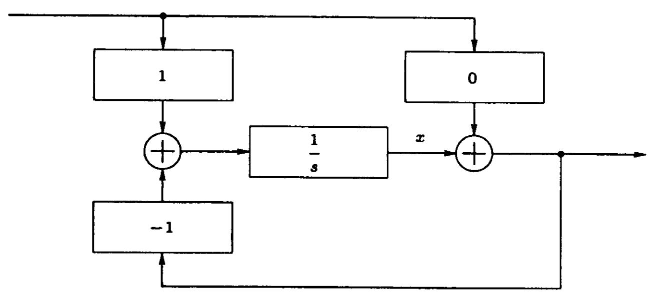
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$

$$y(t) = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

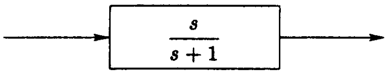
If we use the observer realization form for each of the blocks, then we obtain a slightly different state-space representation.



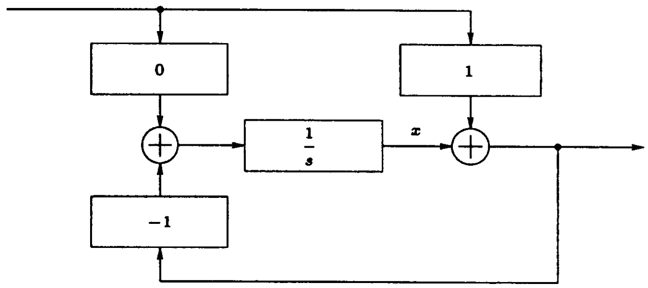

(a) The first gain block.




(b) Observer realization form.

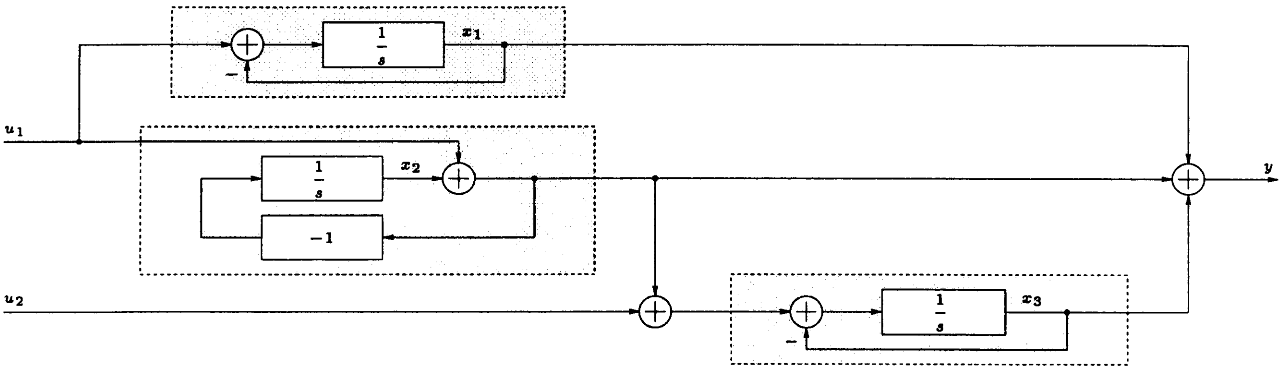



(c) The second gain block.




(d) Observer realization form.

The connected and "expanded" block diagram for this case is shown below.




Similarly, we obtain

$$\dot{x}_1 = -x_1 + u_1,$$
$$\dot{x}_2 = -(x_2 + u_1),$$
$$\dot{x}_3 = -x_3 + \big((x_2 + u_1) + u_2\big),$$

and

$$y = x_1 + (x_2 + u_1) + x_3.$$

And,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$

$$y(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

3. A control system is described in state-space representation, such that

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & -4 & 0 \\ 0 & -2 & 0 \\ -2 & -1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} u(t),$$

where $u$ and $\mathbf{x}$ are the input and the state vectors, respectively.

(a) Determine the poles of the system.

**Solution:** The poles of the system are the eigenvalues of the system. For the state matrix

$$A = \begin{bmatrix} -1 & -4 & 0 \\ 0 & -2 & 0 \\ -2 & -1 & -2 \end{bmatrix},$$

the eigenvalues are obtained from the solution to the characteristic equation $\det(\lambda I - A) = 0$, where $I$ is the appropriately dimensioned identity matrix. In our case,

$$\det(\lambda I - A) = \det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & -4 & 0 \\ 0 & -2 & 0 \\ -2 & -1 & -2 \end{bmatrix}\right)$$

$$= \det \begin{bmatrix} \lambda+1 & 4 & 0 \\ 0 & \lambda+2 & 0 \\ 2 & 1 & \lambda+2 \end{bmatrix}$$

$$= \left((\lambda+1)(\lambda+2)(\lambda+2) + (4)(0)(2) + (0)(0)(1)\right)$$

$$- \left((2)(\lambda+2)(0) + (1)(0)(\lambda+1) + (\lambda+2)(0)(4)\right)$$

$$= (\lambda+1)(\lambda+2)^2 = 0.$$

So, the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -2$, and $\lambda_3 = -2$.

(b) Obtain the solution $x(t)$ for $t \geq 0$; when $x(0) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$, and $u(t) = 1$ for $t \geq 0$.

**Solution:** The solution to the given control system is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau.$$

To determine $e^{At}$, we may use a few different methods. However in this case, the method based on the Cayley-Hamilton theorem is the easiest. In this method, we observe that $e^{At}$ may be described by a linear combination of $A^k$ for $k = 0, \ldots, (n-1)$, so that

$$e^{At} = \alpha_0 I + \alpha_1 A + \ldots + \alpha_{n-1} A^{n-1},$$

where $I$ is the appropriately dimensioned identity matrix, $n$ is the dimension of the system, and $\alpha_0, \ldots, \alpha_{n-1}$ are scalars. The scalars are determined by the application of the eigenvectors to the above equation that results in the set of equations

$$e^{\lambda_1 t} = \alpha_0 + \alpha_1 \lambda_1 + \ldots + \alpha_{n-1} \lambda_1^{n-1},$$

$$\vdots$$

$$e^{\lambda_n t} = \alpha_0 + \alpha_1 \lambda_n + \ldots + \alpha_{n-1} \lambda_n^{n-1},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues. To be able to solve for all the unknown variables $\alpha_0, \ldots, \alpha_n$; we need to have $n$ linearly independent equations. However, when an eigenvalue is repeated, we get the same equation more than once. In such a case, we use the partial derivatives of the equations for the repeated eigenvalue with respect to the eigenvalue, or

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda_i^k}\left(e^{\lambda_i t} = \alpha_0 + \alpha_1\lambda_i + \ldots + \alpha_{n-1}\lambda_i^{n-1}\right)$$

for $k = 1, \ldots, r$, where $r$ is the number of repetitions of the eigenvalue $\lambda_i$. In our case, $n = 3$, so

$$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2;$$

and the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -2$, and $\lambda_3 = -2$. The set of equations becomes

$$\left[e^{\lambda t}\right]_{\lambda=-1} = \left[\,\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\,\right]_{\lambda=-1},$$

$$\left[e^{\lambda t}\right]_{\lambda=-2} = \left[\,\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\,\right]_{\lambda=-2},$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(e^{\lambda t}\right)\right]_{\lambda=-2} = \left[\,\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right)\,\right]_{\lambda=-2}.$$

In our case,

$$\begin{aligned} e^{(-1)t} &= \alpha_0 + \alpha_1(-1) + \alpha_2(-1)^2, \\ e^{(-2)t} &= \alpha_0 + \alpha_1(-2) + \alpha_2(-2)^2, \\ te^{(-2)t} &= \alpha_1 + 2\alpha_2(-2); \end{aligned}$$

or

$$\begin{aligned} e^{-t} &= \alpha_0 - \alpha_1 + \alpha_2, \\ e^{-2t} &= \alpha_0 - 2\alpha_1 + 4\alpha_2, \\ te^{-2t} &= \alpha_1 - 4\alpha_2. \end{aligned}$$

Solving the above set of equations simultaneously gives

$$\begin{aligned} \alpha_0 &= 4e^{-t} - (3+2t)e^{-2t}, \\ \alpha_1 &= 4e^{-t} - (4+3t)e^{-2t}, \\ \alpha_2 &= e^{-t} - (1+t)e^{-2t}. \end{aligned}$$

As a result,

$$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

$$= \left(4e^{-t} - (3+2t)e^{-2t}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \left(4e^{-t} - (4+3t)e^{-2t}\right)\begin{bmatrix} -1 & -4 & 0 \\ 0 & -2 & 0 \\ -2 & -1 & -2 \end{bmatrix}$$

$$+ \left(e^{-t} - (1+t)e^{-2t}\right)\begin{bmatrix} 1 & 12 & 0 \\ 0 & 4 & 0 \\ 6 & 12 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & -4e^{-t} + 4e^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ -2e^{-t} + 2e^{-2t} & 8e^{-t} - (8+9t)e^{-2t} & e^{-2t} \end{bmatrix}.$$

Once $e^{At}$ is determined, we have

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$= \begin{bmatrix} e^{-t} & -4e^{-t} + 4e^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ -2e^{-t} + 2e^{-2t} & 8e^{-t} - (8+9t)e^{-2t} & e^{-2t} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$+ \int_0^t \begin{bmatrix} e^{-(t-\tau)} & -4e^{-(t-\tau)} + 4e^{-2(t-\tau)} & 0 \\ 0 & e^{-2(t-\tau)} & 0 \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & 8e^{-(t-\tau)} - \left(8 + 9(t-\tau)\right)e^{-2(t-\tau)} & e^{-2(t-\tau)} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} d\tau$$

$$= \begin{bmatrix} -4e^{-t} + 4e^{-2t} \\ e^{-2t} \\ 8e^{-t} - (7+9t)e^{-2t} \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} \\ 0 \\ -2e^{-(t-\tau)} + e^{-2(t-\tau)} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} -4e^{-t} + 4e^{-2t} \\ e^{-2t} \\ 8e^{-t} - (7+9t)e^{-2t} \end{bmatrix} + \begin{bmatrix} e^{-(t-\tau)} \\ 0 \\ -2e^{-(t-\tau)} + (1/2)e^{-2(t-\tau)} \end{bmatrix}_{\tau=0}^{\tau=t}$$

$$= \begin{bmatrix} -4e^{-t} + 4e^{-2t} \\ e^{-2t} \\ 8e^{-t} - (7+9t)e^{-2t} \end{bmatrix} + \begin{bmatrix} 1 - e^{-t} \\ 0 \\ -2\left(1 - e^{-t}\right) + (1/2)\left(1 - e^{-2t}\right) \end{bmatrix}$$

Therefore,

$$\mathbf{x}(t) = \begin{bmatrix} 1 - 5e^{-t} + 4e^{-2t} \\ e^{-2t} \\ -3/2 + 10e^{-t} - (15/2 + 9t)e^{-2t} \end{bmatrix} \text{ for } t \geq 0.$$

4. A control system is described in state-space representation, such that

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} -2 & 1 \end{bmatrix} \mathbf{x}(t),$$

where $u$, $\mathbf{x}$, and $y$ are the input, the state, and the output variables, respectively. Determine the transfer function or the transfer matrix of the system.

**Solution:** The transfer matrix of a control system described in the state-state representation

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t),$$

is

$$F(s) = C(sI - A)^{-1}B + D,$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$C = \begin{bmatrix} -2 & 1 \end{bmatrix}, \qquad D = 0,$$

and $I$ is the appropriately dimensioned identity matrix. So,

$$F(s) = \begin{bmatrix} -2 & 1 \end{bmatrix} \left( s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0$$

$$= \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{(s+2)^2} \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{(s+2)^2} \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} s+2 \\ 2(s+2) \end{bmatrix}$$

$$= \frac{1}{(s+2)^2}(0).$$

In other words, the transfer function is $F(s) = 0$.

5. A time-varying control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \sin(t) & \cos(t) \\ 0 & \sin(t) \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{x}$ is the state variable. Determine $\mathbf{x}(2\pi)$, when $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$.

**Solution:** In this case, the state matrix

$$A(t) = \begin{bmatrix} \sin(t) & \cos(t) \\ 0 & \sin(t) \end{bmatrix}$$

satisfies the commutativity condition $A(t_1)A(t_2) = A(t_2)A(t_1)$ for all $t_1$ and $t_2$, since

$$\begin{aligned} A(t_1)A(t_2) &= \begin{bmatrix} \sin(t_1) & \cos(t_1) \\ 0 & \sin(t_1) \end{bmatrix} \begin{bmatrix} \sin(t_2) & \cos(t_2) \\ 0 & \sin(t_2) \end{bmatrix} \\ &= \begin{bmatrix} \sin(t_1)\sin(t_2) & \sin(t_1)\cos(t_2) + \cos(t_1)\sin(t_2) \\ 0 & \sin(t_1)\sin(t_2) \end{bmatrix} \\ &= \begin{bmatrix} \sin(t_2)\sin(t_1) & \sin(t_2)\cos(t_1) + \cos(t_2)\sin(t_1) \\ 0 & \sin(t_2)\sin(t_1) \end{bmatrix} \\ &= A(t_2)A(t_1). \end{aligned}$$

As a result, the solution for the state variable is given by

$$\mathbf{x}(t) = e^{\left(\int_0^t A(\tau)\,d\tau\right)}\mathbf{x}(0).$$

So,

$$\mathbf{x}(2\pi) = e^{\left(\int_0^{2\pi} \begin{bmatrix} \sin(\tau) & \cos(\tau) \\ 0 & \sin(\tau) \end{bmatrix} d\tau\right)}\mathbf{x}(0) = e^{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}\mathbf{x}(0) = \mathbf{x}(0),$$

or

$$\mathbf{x}(2\pi) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T.$$