EE 435 **Exam#1** 75 minutes **Nov. 12, 1996**

1. Given the set of vectors,

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

determine an orthogonal set spanning the same space using the Gram-Schmidt orthogonalization procedure. (10pts)

2. Consider the following system of linear equations.

$$\begin{bmatrix} 2 & 1 & -2 & 0 & -4 & -1 \\ 1 & 6 & -1 & 3 & 0 & -3 \\ -5 & 0 & 4 & 0 & -1 & 0 \\ 3 & 5 & 0 & -2 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 4 \\ 0 \end{bmatrix}.$$

Determine and justify whether or not this system of equations has a solution *without* finding an explicit solution. (10pts)

3. Consider the two state-space descriptions,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{x}(t),$$

and

$$\dot{\mathbf{z}}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{z}(t)$$

where $\mathbf{x}$ and $\mathbf{z}$ are the state variables. Determine whether or not these two descriptions represent the same control system. (10pts)

4. The dynamics of a control system are described by

$$\ddot{y}(t) + 4\dot{y}(t) + 4y(t) = \dot{u}(t) + u(t),$$

where u and y are the input and the output variables, respectively.

(a) Realize the system using one of the canonical forms, and obtain a state-space representation based on that canonical form. (15pts)

(b) Determine the new basis vectors that transform the system into a diagonal or a Jordan form, and obtain the transformed system description. (15pts)

5. The dynamics of a control system are described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -9 & -12 \\ 6 & 8 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(t),$$

where u and $\mathbf{x}$ are the input and the state variables, respectively.

(a) Determine $\mathbf{x}(t)$ for $t \geq 0$, when $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, and $u(t) = 0$ for $t \geq 0$. (15pts)

(b) Determine $\mathbf{x}(t)$ for $t \geq 0$, when $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, and $u(t) = 1$ for $t \geq 0$. (15pts)

(c) Determine whether the system is marginally or asymptotically stable, when $u(t) = 0$; and whether or not it is bounded-input bounded-output stable, when $u(t) \neq 0$. (10pts)

22-141 50 SHEETS
22-142 100 SHEETS
22-144 200 SHEETS
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#1

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ and } x_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

by observation $x_2 = x_1 + x_3$

or $x_3 = x_2 - x_1$

so there will only be two nonzero vectors

let $\hat{x}_1 = x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$x_2' = x_2 - \frac{\langle x_2, \hat{x}_1 \rangle}{\langle \hat{x}_1, \hat{x}_1 \rangle} \hat{x}_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \frac{(2)(1)+(1)(0)+(1)(1)}{(1)^2+(0)^2+(1)^2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

so let $\hat{x}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$\Rightarrow$ One set of orthogonal vectors spanning the same space is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

#2

$$\underbrace{\begin{bmatrix} \cdots & -2 & \cdots \\ & -1 & \\ & 4 & \\ & 0 & \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_6 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ -1 \\ 4 \\ 0 \end{bmatrix}}_{B}$$

By inspection, we realize that a column of A is ~~three~~ same as B, i.e.

$$\operatorname{rank}([A \mid B]) = \operatorname{rank}(A)$$

$\Rightarrow$ There is at least one solution

#3

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x = Ax$$

$$\dot{z} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z = Bz$$

Assume a similarity transformation

$$T^{-1}AT = B$$

or $\quad AT = TB$

let $T = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix}$

$$AT = TB \Rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(1,1) term $\quad t_3 = t_1$

(1,2) term $\quad t_4 = t_2$

(2,1) term $\quad -t_1 + 2t_3 = t_3 \longrightarrow t_3 = t_3$ no information

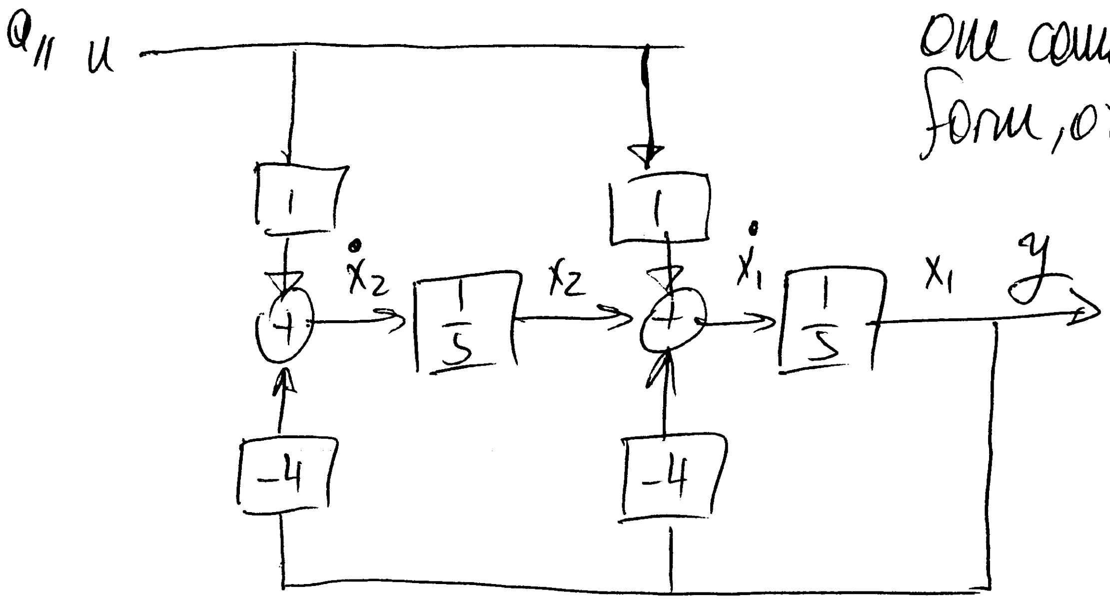
(2,2) term $\quad -t_2 + 2t_4 = t_4 \longrightarrow t_4 = t_4$ " "

so $T = \begin{bmatrix} t_1 & t_2 \\ t_1 & t_2 \end{bmatrix}$

but T is singular $\Rightarrow$ T is not a valid transformation
i.e. There is no such transformation

Two systems are different.

#4 $\quad \ddot{y} + 4\dot{y} + 4y = \dot{u} + u$



One canonical form, other possible

From the realization

$$\dot{x}_1 = -4x_1 + x_2 + u \qquad ; \quad y = x_1$$
$$\dot{x}_2 = -4x_1 + u$$

or
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b// The eigenvalues from

$$\det(\lambda I - A) = 0$$

$$\det \begin{bmatrix} \lambda+4 & -1 \\ 4 & \lambda \end{bmatrix} = (\lambda+4)\lambda - (-1)(4)$$
$$= \lambda^2 + 4\lambda + 4$$
$$= (\lambda+2)^2$$
$$= 0 \Rightarrow \lambda_{1,2} = -2$$

The eigenvectors from

$$(A - \lambda I)u = 0$$

$$\left(\begin{bmatrix} -4 & 1 \\ -4 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\lambda = -2 \Rightarrow \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

rank drop = 1 so only 1 eigenvector

$$\left.\begin{array}{l} -2u_{1,1} + u_{2,1} = 0 \\ -4u_{1,1} + 2u_{2,1} = 0 \end{array}\right\} \Rightarrow u_{2,1} = 2u_{1,1}$$

let $u_{1,1} = 1 \Rightarrow u_{2,1} = 2$ or $\begin{bmatrix} u_{1,1} \\ u_{2,1} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

To find another vector, we use the generalized eigenvalue problem which will result in a Jordan form.

So $(A - \lambda I)u_2 = u_1$

$$\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} u_{2,1} \\ u_{2,2} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\left.\begin{array}{l} -2u_{2,1} + u_{2,2} = 1 \\ -4u_{2,1} + 2u_{2,2} = 2 \end{array}\right\} \Rightarrow u_{2,2} = 1 + 2u_{2,1}$$

let $u_{2,1} = 1 \Rightarrow u_{2,2} = 3 \Rightarrow \begin{bmatrix} u_{2,1} \\ u_{2,2} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\text{So } T = \begin{bmatrix} u_{1,1} & u_{2,1} \\ u_{2,1} & u_{2,2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

let $x = Tz \Rightarrow \dot{z} = T^{-1}ATz + T^{-1}Bu$

$$y = CTz$$

$$T^{-1}AT = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \leftarrow \text{Jordan Form}$$

$$T^{-1}B = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$CT = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

So $\dot{z} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} z + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} z$$

#5

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -9 & -12 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

(a) let $u = 0$, $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$x(t) = e^{At} x(0)$$

To determine e^{At}, we need to find the eigenvalues

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda+9 & 12 \\ -6 & \lambda-8 \end{bmatrix}$$

$$= (\lambda+9)(\lambda-8) - (-6)(12)$$

$$= \lambda^2 + \lambda = \lambda(\lambda+1)$$

$$= 0 \Rightarrow \lambda = 0, -1$$

so $e^{At} = a_0 I + a_1 A$

and $e^{\lambda t} = a_0 + a_1 \lambda \xrightarrow{\lambda = 0} 1 = a_0$

$$\xrightarrow{\lambda=-1} e^{-t} = a_0 - a_1$$

$$\Rightarrow a_1 = 1 - e^{-t}$$

$$e^{At} = I + (1-e^{-t})A$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (1-e^{-t}) \begin{bmatrix} -9 & -12 \\ 6 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1-9(1-e^{-t}) & -12(1-e^{-t}) \\ 6(1-e^{-t}) & 1+8(1-e^{-t}) \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} -8+9e^{-t} & -12+12e^{-t} \\ 6-6e^{-t} & 9-8e^{-t} \end{bmatrix}$$

$$x(t) = e^{At}x(0) = \begin{bmatrix} -12+12e^{-t} \\ 9-8e^{-t} \end{bmatrix}$$

(b) $u(t) = 1$, $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$= \int_0^t \begin{bmatrix} -8+9e^{-(t-\tau)} & -12+12e^{-(t-\tau)} \\ 6-6e^{-(t-\tau)} & 9-8e^{-(t-\tau)} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} d\tau$$

$$= \int_0^t \begin{bmatrix} -4+3e^{-(t-\tau)} \\ 3-2e^{-(t-\tau)} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} -4\tau+3e^{-(t-\tau)} \\ 3\tau-2e^{-(t-\tau)} \end{bmatrix}_{\tau=0}^{t}$$

$$= \begin{bmatrix} -4t+3-3e^{-t} \\ 3t-2+2e^{-t} \end{bmatrix}$$

So $x(t) = \begin{bmatrix} -12+12e^{-t} \\ 9-8e^{-t} \end{bmatrix} + \begin{bmatrix} -4t+3-3e^{-t} \\ 3t-2+2e^{-t} \end{bmatrix}$

$$= \begin{bmatrix} -9+4t+9e^{-t} \\ 7+3t-6e^{-t} \end{bmatrix}$$

(c) $u(t)=0$ eigenvalues $\lambda = 0, -1$

since there is an eigenvalue on the imaginary axis and it is not repeated, the system is stable but not asymptotically stable.

We can see that it is not asymptotically stable from part (a) as well since

$$\lim_{t\to\infty} x(t) = \begin{bmatrix} -12 \\ 9 \end{bmatrix} \neq 0$$

$u(t) \neq 0$

the eigenvalue on the imaginary axis is a problem, since if we have an input with the same eigenvalue (or pole) then the poles will repeat on the imaginary axis and the system will be unstable, as can be seen from part (b) as well; since

$$\lim_{t\to\infty} x(t) = \begin{bmatrix} \infty \\ \infty \end{bmatrix} \text{ for } u(t) = 1$$

which is bounded.

Therefore, the system is not bounded-input bounded-output stable.