EE 431 | Exam#1 | Nov. 11, 1997

75 minutes

1. A control system is described in state-space representation, such that

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -3 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} u(t),$$

where $\mathbf{x}$ and u are the state and the input variables, respectively.

   (a) Determine the poles of the system. (05pts)

   (b) Determine the linear transformation that would put it into the diagonal or Jordan form. Express the new state-space equation that describes the system. (20pts)

   (c) Obtain the solution $\mathbf{x}(t)$ for $t \geq 0$; when $\mathbf{x}(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, and $u(t) = 0$. (10pts)

2. Determine $\mathbf{x}(100)$ for the following discrete-time control systems, when $\mathbf{x}(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

   (a) $\mathbf{x}(k+1) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}(k).$ (10pts)

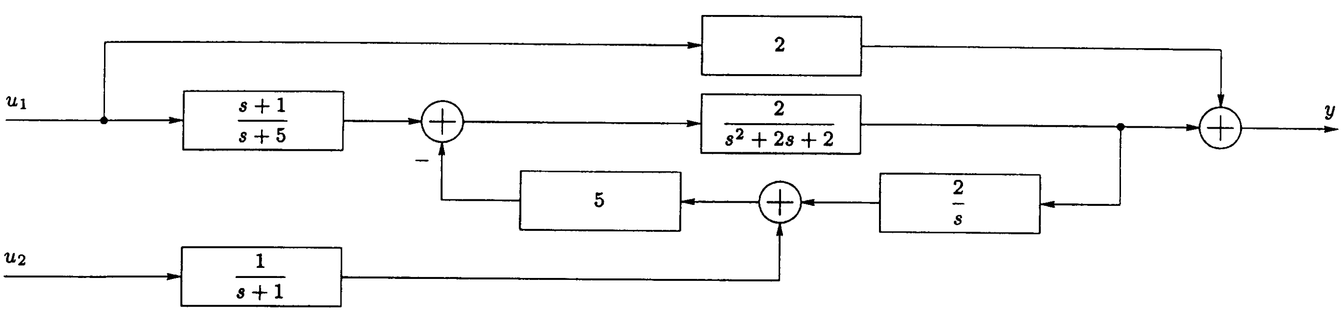
   (b) $\mathbf{x}(k+1) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x}(k).$ (15pts)

3. A time-varying control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -e^{-t} & 0 \\ 0 & -1 - \cos(t) \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{x}$ is the state variable. Determine $\mathbf{x}(5)$, when $\mathbf{x}(0) = \begin{bmatrix} 10 & 20 \end{bmatrix}^T$. (15pts)

4. The block diagram of a control system is given below.



Obtain a state-space representation of the system without any block-diagram reduction. (25pts)

#1

$$\dot{x} = \underbrace{\begin{bmatrix} -3 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & -3 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{B} u$$

(a) Poles or the eigenvalues from the solution of the characteristic equation.

$$\det(\lambda I - A) = 0$$

$$\det \begin{bmatrix} \lambda+3 & 0 & -2 \\ 0 & \lambda+1 & 0 \\ -2 & 0 & \lambda+3 \end{bmatrix} = 0$$

$$[(\lambda+3)(\lambda+1)(\lambda+3)+0+0] - [(-2)(\lambda+1)(-2)+0+0] = 0$$

$$(\lambda+1)[(\lambda+3)^2 - 4] = 0$$

$$(\lambda+1)(\lambda^2+6\lambda+5) = 0$$

$$(\lambda+1)(\lambda+1)(\lambda+5) = 0 \Rightarrow \lambda = -1, -1, -5$$

(b) The transformation $x = Tz$ is when

$T = \begin{bmatrix} u_1 & u_2 & u_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$ where u_i's are the eigenvectors

Moreover, since A is symmetric, T can be chosen orthonormal.

$$(A - \lambda I)u = 0$$

or $(\lambda I - A)u = 0 \Rightarrow \begin{bmatrix} \lambda+3 & 0 & -2 \\ 0 & \lambda+1 & 0 \\ -2 & 0 & \lambda+3 \end{bmatrix} \begin{bmatrix} u_{1i} \\ u_{2i} \\ u_{3i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\lambda=-1 \Rightarrow \begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2u_{11} - 2u_{31} = 0$ is the only eqn.

$\Rightarrow u_{11} = u_{31}$

let $u_{11} = 1 \Rightarrow u_{31} = 1 \Rightarrow u_1 = \begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix}$

another choice let $u_{11} = 0 \Rightarrow u_{31} = 0 \Rightarrow u_2 = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$

Choose $\langle u_1, u_2 \rangle = 0$

$\langle u_1, u_2 \rangle = (1)(0) + a \cdot b + (1)(0)$

$= ab$

$\Rightarrow a = 0$ or $b = 0$

since $b=0 \Rightarrow u_2 = 0$ not a valid eigenvector

so $a=0$ and let $b=1$

$$\Rightarrow u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{\text{normalize}} u_1^* = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longrightarrow u_2^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda=-5 \Rightarrow \begin{bmatrix} -2 & 0 & -2 \\ 0 & -4 & 0 \\ -2 & 0 & -2 \end{bmatrix}\begin{bmatrix} u_{13} \\ u_{23} \\ u_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2u_{13} - 2u_{33} = 0 \Rightarrow u_{33} = -u_{13}$

$-4u_{23} = 0 \Rightarrow u_{23} = 0$

$$\text{let } u_{13} = 1 \Rightarrow u_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \longrightarrow u_3^* = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{bmatrix}$$

$$\text{or } T = \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix}$$

$$\text{let } x = Tz \Rightarrow \dot{z} = T^T A T z + T^T B u$$

$$= \begin{bmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -5 \end{bmatrix} z + \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} u$$

$$\dot{z} = \begin{bmatrix} -1 & & \\ & -1 & \\ & & -5 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \end{bmatrix} u$$

(c) $x(t) = e^{At} x(0)$ when $u(t) = 0$

$$= T e^{T^T A T t} T^t x(0)$$

$$= T e^{\Lambda_A t} T^t x(0)$$

$$= \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} e^{-t} & & \\ & e^{-t} & \\ & & e^{-5t} \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} e^{-t} & & \\ & e^{-t} & \\ & & e^{-5t} \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2\, e^{-t} \\ 0 \\ \sqrt{2}/2\, e^{-5t} \end{bmatrix} = \begin{bmatrix} 1/2\, e^{-t} + 1/2\, e^{-5t} \\ 0 \\ 1/2\, e^{-t} - 1/2\, e^{-5t} \end{bmatrix}$$

#2 $x(k+1) = A x(k) \Rightarrow x(k) = A^k x(0)$

$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(a) $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$x(100) = A^{100} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (-1)^{100} & 0 \\ 0 & (-1)^{100} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \longleftarrow \lambda = -1, -1$ since upperdiagonal

$$x(100) = A^{100} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$A^{100} = aI + bA$ and $\lambda^{100} = a + b\lambda$

$\frac{d}{d\lambda}(\lambda^{100} = a + b\lambda)$ since λ is repeated

or $\left.\begin{matrix} \lambda^{100} = a + b\lambda \\ 100\lambda^{99} = b \end{matrix}\right\} \lambda = -1 \quad \left\{\begin{matrix} (-1)^{100} = a + b(-1) \\ 100(-1)^{99} = b \end{matrix}\right.$

$$\Rightarrow b = -100 \Rightarrow a = b + 1 = -99$$

$$A^{100} = -99I - 100A$$

$$= \begin{bmatrix} -99 & 0 \\ 0 & -99 \end{bmatrix} + \begin{bmatrix} 100 & -100 \\ 0 & 100 \end{bmatrix} = \begin{bmatrix} 1 & -100 \\ 0 & 1 \end{bmatrix}$$

$$x(100) = \begin{bmatrix} 1 & -100 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -99 \\ 1 \end{bmatrix}$$

#3

$$\dot{x} = \underbrace{\begin{bmatrix} -e^{-t} & 0 \\ 0 & -1-\cos t \end{bmatrix}}_{A(t)} x \quad , \quad x(0) = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$x(t) = e^{\int_0^t A(\tau)d\tau} \cdot x(0)$$

$$= e^{\begin{bmatrix} -\int_0^t e^{-\tau}d\tau & 0 \\ 0 & -\int_0^t (1+\cos\tau)d\tau \end{bmatrix}} \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

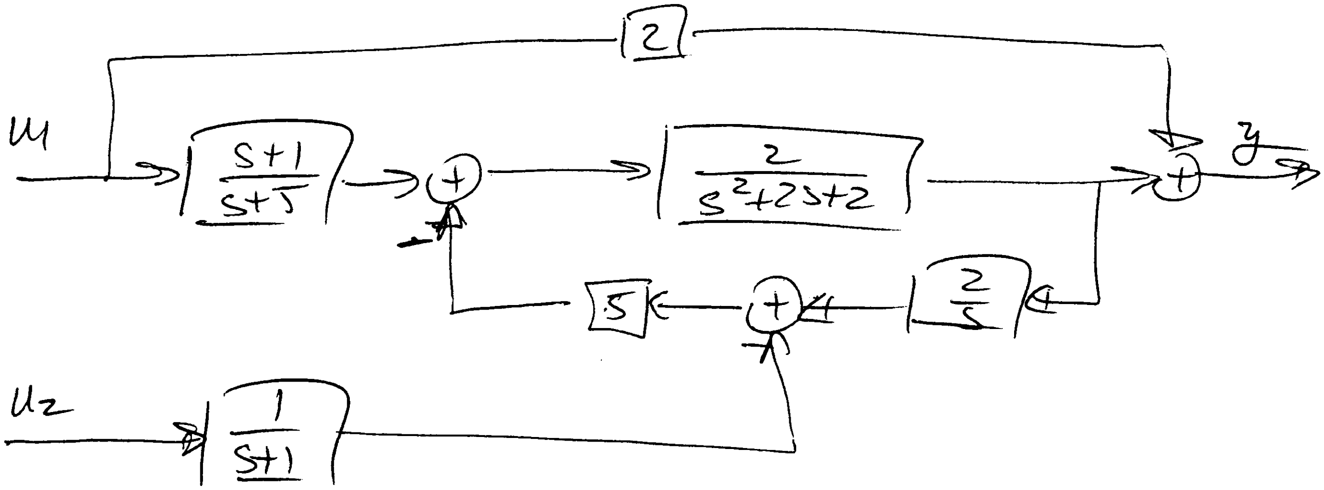
$$= e^{\begin{bmatrix} e^{-t}-1 & 0 \\ 0 & -(t+\sin t)+(0+0) \end{bmatrix}} \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} e^{e^{-t}-1} & 0 \\ 0 & e^{-t-\sin t} \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 10\,e^{e^{-t}-1} \\ 20\,e^{-t-\sin t} \end{bmatrix}$$

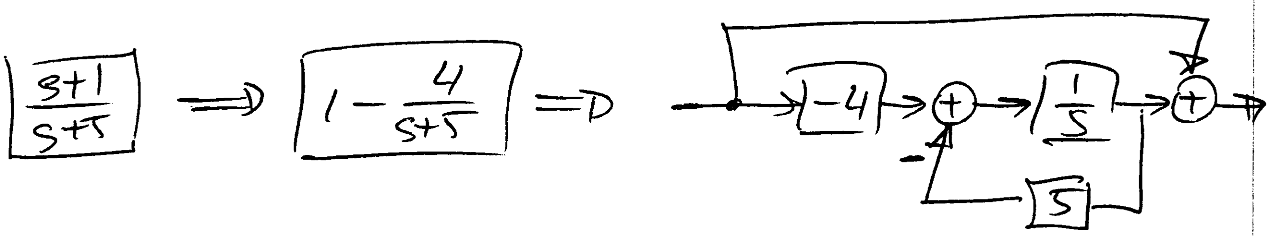
$$x(5) = \begin{bmatrix} 10e^{e^{-5}-1} \\ 20\,e^{-5-\sin 5} \end{bmatrix} = \begin{bmatrix} 3.7037 \\ 0.3516 \end{bmatrix}$$

in radians

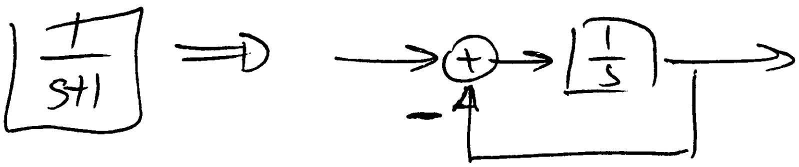
#4



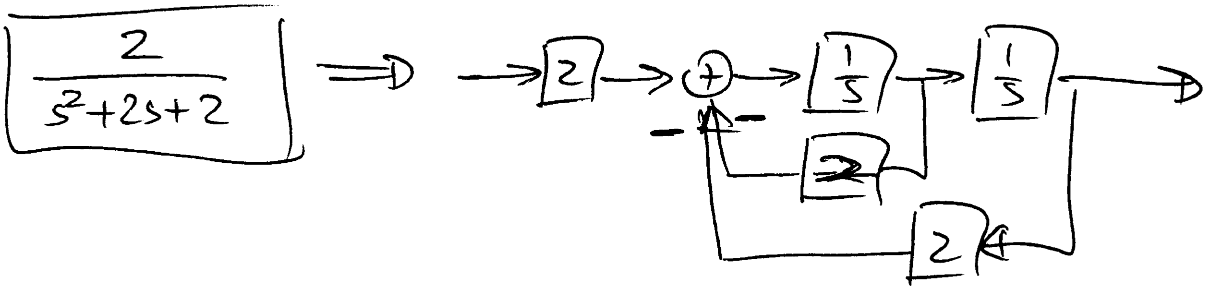
u_1
$\frac{s+1}{s+5}$
$\frac{2}{s^2+2s+2}$
2
y
$\frac{2}{s}$
5
u_2
$\frac{1}{s+1}$



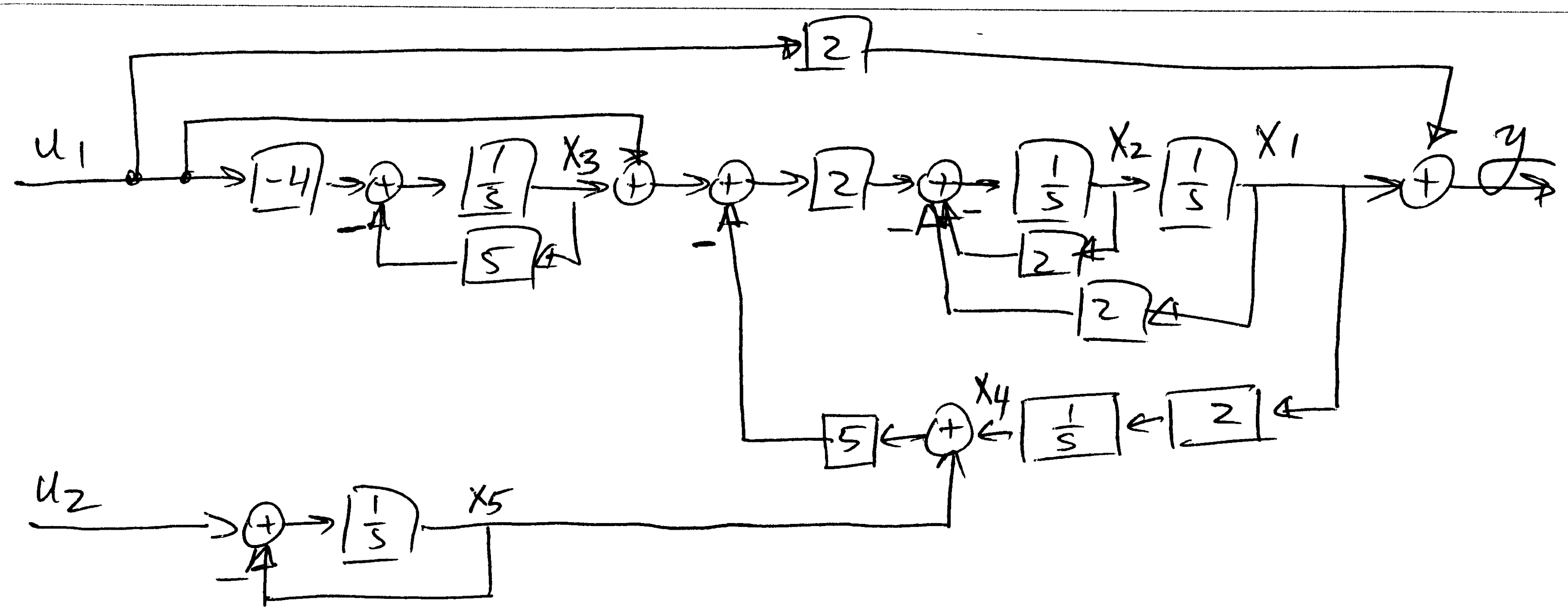
$\frac{s+1}{s+5}$
$1-\frac{4}{s+5}$
-4
$\frac{1}{s}$
5



$\frac{1}{s+1}$
$\frac{1}{s}$



$\frac{2}{s^2+2s+2}$
2
$\frac{1}{s}$
$\frac{1}{s}$
2
2



$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_2 - 2x_1 + 2\left[(u_1 + x_3) - 5(x_4 + x_5)\right]$$

$$= -2x_1 - 2x_2 + 2x_3 - 10x_4 - 10x_5 + 2u_1$$

$$\dot{x}_3 = -5x_3 - 4u_1$$

$$\dot{x}_4 = 2x_1$$

$$\dot{x}_5 = -x_5 + u_2$$

$$y = x_1 + 2u_1$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -2 & -2 & 2 & -10 & -10 \\ 0 & 0 & -5 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ -4 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$