EE 431

Exam#2
75 minutes

Dec. 04, 2003

1. Consider the autonomous linear control system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{x}$ is the state variable. Obtain a Lyapunov function to prove its stability or instability. (25pts)

2. A time-varying control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \sin(t) & \cos(t) \\ 0 & \sin(t) \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{x}$ is the state variable. Determine whether or not the initial condition $\mathbf{x}(0)$ can be obtained from the output $y(t)$ for $0 \le t \le \pi/2$, where

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t).$$

(25pts)

3. A control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t),$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Obtain its Kalman decomposition that separates the controllable, uncontrollable, observable, and unobservable portions. Clearly mark the portions on the decomposed system. (25pts)

4. The transfer matrix of a control system is given by

$$H(s) = \begin{bmatrix} \dfrac{s+2}{(s+1)(s+3)} & \dfrac{s+2}{(s+1)^2(s+3)} \\ \dfrac{1}{s(s+3)} & \dfrac{3s+7}{s(s+1)^2(s+3)} \\ \dfrac{2/3}{(s+1)(s+3)} & \dfrac{1}{s(s+1)(s+3)} \end{bmatrix}.$$

Obtain its left or right coprime factorization. Determine the order of a controllable and observable realization of the system. (25pts)

1. Consider the autonomous linear control system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{x}$ is the state variable. Obtain a lyapunov function to prove its stability or instability.

Solution: A lyapunov function for a linear system is $L(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$; such that if a positive definite and symmetric matrix P satisfies the lyapunov equation

$$A^T P + PA = -Q,$$

for the state matrix A and a positive definite and symmetric matrix Q, then the system is asymptotically stable. Note that this result is directly obtained from the relation

$$\frac{\mathrm{d}L(\mathbf{x})}{\mathrm{d}t} = \frac{\mathrm{d}(\mathbf{x}^T P \mathbf{x})}{\mathrm{d}t} = \dot{\mathbf{x}}^T P \mathbf{x} + \mathbf{x}^T P \dot{\mathbf{x}} = (A\mathbf{x})^T P \mathbf{x} + \mathbf{x}^T P (A\mathbf{x}) = \mathbf{x}^T (A^T P + PA)\mathbf{x}$$
$$= -\mathbf{x}^T Q \mathbf{x} < 0 \text{ for all } \mathbf{x} \neq \mathbf{0}.$$

In our case

$$A = \begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix}.$$

Since any positive definite and symmetric matrix Q should work, let Q be the 2×2 identity matrix, and

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}.$$

Checking for a solution to the lyapunov equation, we get

$$\begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix}^T \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

or

$$\begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Due to the symmetry, we get three equations from the matrix equation.

$$-2p_1 - 2p_2 = -1,$$

$$2p_1 - 5p_2 - p_3 = 0,$$

and

$$4p_2 - 8p_3 = -1.$$

From the first and the third equations, we get $p_1 = (1-2p_2)/2$, and $p_3 = (1+4p_2)/8$. Substituting these expressions into the second equation, we get

$$2\left(\frac{1-2p_2}{2}\right) - 5p_2 - \left(\frac{1+4p_2}{8}\right) = 0,$$

$$1 - 2p_2 - 5p_2 - 1/8 - (1/2)p_2 = 0,$$

or $p_2 = 7/60$. As a result, we get

$$P = (1/60)\begin{bmatrix} 23 & 7 \\ 7 & 11 \end{bmatrix}.$$

From the positiveness of the principal minors of P, $p_1 = (23/60) > 0$ (or $p_3 = (11/60) > 0$), and $\det(P) = p_1p_3 - p_2{}^2 = (204/60) > 0$; we conclude that P is positive definite. Since there exists a lyapunov function

$$L(\mathbf{x}) = (1/60)\,\mathbf{x}^T \begin{bmatrix} 23 & 7 \\ 7 & 11 \end{bmatrix} \mathbf{x} > 0,$$

such that

$$\frac{\mathrm{d}L(\mathbf{x})}{\mathrm{d}t} = -\mathbf{x}^T\mathbf{x} < 0 \text{ for all } \mathbf{x} \neq \mathbf{0};$$

the system is asymptotically stable.

2. A time-varying control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \sin(t) & \cos(t) \\ 0 & \sin(t) \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{x}$ is the state variable. Determine whether or not the initial condition $\mathbf{x}(0)$ can be obtained from the output $y(t)$ for $0 \le t \le \pi/2$, where

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t).$$

Solution: The property of determining the initial condition from the future values of the output is the observability property. To ensure observability of the system, the rank of the observability matrix should be full. However, in this case the system is time varying, and we need to check the rank of the observability grammian, which is given by

$$\mathfrak{O}(t_0, t_1) = \int_{t_0}^{t_1} \Phi^t(t, t_0) C^T(t) C(t) \Phi(t, t_0)\, \mathrm{d}t,$$

where Φ and C are the state-transition and the output matrices, respectively. Since the state matrix

$$A(t) = \begin{bmatrix} \sin(t) & \cos(t) \\ 0 & \sin(t) \end{bmatrix}$$

satisfies the commutativity condition $A(t_1)A(t_2) = A(t_2)A(t_1)$ for all t_1 and t_2, the state-transition matrix is given by

$$\Phi(t,0) = e^{\left(\int_0^t \begin{bmatrix} \sin(\tau) & \cos(\tau) \\ 0 & \sin(\tau) \end{bmatrix} \mathrm{d}\tau\right)} = e^{\left(\begin{bmatrix} -\cos(\tau) & \sin(\tau) \\ 0 & -\cos(\tau) \end{bmatrix}_{\tau=0}^{\tau=t}\right)} = e^{\begin{bmatrix} 1-\cos(t) & \sin(t) \\ 0 & 1-\cos(t) \end{bmatrix}};$$

or

$$\Phi(t,0) = \begin{bmatrix} e^{1-\cos(t)} & \phi_{1,2}(t) \\ 0 & e^{1-\cos(t)} \end{bmatrix},$$

since the exponent matrix is upper triangular.

REMARK: In this case, we really don't need to determine the $\phi_{1,2}(t)$ term, because it disappears in the product $C(t)\Phi(t,0)$. However, if we decide to determine it, we may do so by using a few different methods. For demonstration purposes, we will use the Cayley-Hamilton theorem, where $\Phi(t,0)$ may be described by a linear combination of the powers of the exponent matrix up to one less than the dimension of the exponent matrix. In our case,

$$\Phi(t,0) = \alpha_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1-\cos(t) & \sin(t) \\ 0 & 1-\cos(t) \end{bmatrix},$$

where α_0 and α_1 are scalars, since the dimension of the matrix is two. The scalars are determined by the application of the eigenvectors to the above equation that results in a set of equations, where eigenvalues replace the exponent matrices. Both of the eigenvalues of the exponent matrix are $1-\cos(t)$ in our case. When an eigenvalue is repeated, we get the same equation more than once. In such a case, we use the partial derivatives of the equations for the repeated eigenvalue with respect to the eigenvalue. In our case, the set of equations becomes

$$e^{1-\cos(t)} = \alpha_0(1) + \alpha_1(1-\cos(t))$$
$$e^{1-\cos(t)} = \alpha_1(1).$$

Solving the above set of equations simultaneously gives

$$\alpha_0 = \cos(t)e^{1-\cos(t)}$$
$$\alpha_1 = e^{1-\cos(t)}.$$

As a result,

$$\Phi(t,0) = \cos(t)e^{1-\cos(t)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{1-\cos(t)} \begin{bmatrix} 1-\cos(t) & \sin(t) \\ 0 & 1-\cos(t) \end{bmatrix}$$
$$= \begin{bmatrix} e^{1-\cos(t)} & \sin(t)e^{1-\cos(t)} \\ 0 & e^{1-\cos(t)} \end{bmatrix}.$$

The observability grammian becomes

$$\mathfrak{O}(0,t_1) = \int_0^{t_1} \begin{bmatrix} e^{1-\cos(t)} & \phi_{1,2}(t) \\ 0 & e^{1-\cos(t)} \end{bmatrix}^T \begin{bmatrix} 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e^{1-\cos(t)} & \phi_{1,2}(t) \\ 0 & e^{1-\cos(t)} \end{bmatrix} dt$$
$$= \int_0^{t_1} \begin{bmatrix} e^{1-\cos(t)} & 0 \\ \phi_{1,2}(t) & e^{1-\cos(t)} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{1-\cos(t)} & \phi_{1,2}(t) \\ 0 & e^{1-\cos(t)} \end{bmatrix} dt$$
$$= \int_0^{t_1} \begin{bmatrix} 0 & 0 \\ 0 & e^{2(1-\cos(t))} \end{bmatrix} dt = \begin{bmatrix} 0 & 0 \\ 0 & \int_0^{t_1} e^{2(1-\cos(t))}\, dt \end{bmatrix}.$$

The observability grammian

$$\mathfrak{O}(0,t_1) = \begin{bmatrix} 0 & 0 \\ 0 & \int_0^{t_1} e^{2(1-\cos(t))}\, dt \end{bmatrix}$$

is singular, therefore the initial condition cannot be uniquely determined from the future values of the output.

3. A control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t),$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Obtain its kalman decomposition that separates the controllable, uncontrollable, observable, and unobservable portions. Clearly mark the portions on the decomposed system.

Solution: The given system is in block-diagonal form with one 2×2 jordan block and a diagonal element. In this form, we may be able to determine the controllability and the observability of each state by inspection.

In order for all the eigenvalues in a jordan block to be controllable, the input matrix has to have a non-zero element for the last eigenvalue in the jordan block. In our case, the second element of the input matrix is zero which implies that at least the second eigenvalue in the jordan block is not controllable. (If the second-element was non-zero, then the first and the second eigenvalues in the jordan block would have been controllable.) However, because the first element of the input matrix is non-zero, the first eigenvalue in the jordan block is controllable. The distinct eigenvalue is also controllable, since the third element of the input matrix is non-zero. As a result, we observed that the first and the third state variables are controllable, and the second state variable is not controllable.

Similarly, in order for all the eigenvalues in a jordan block to be observable, the output matrix has to have a non-zero element for the first eigenvalue in the jordan block. In our case, the first element of the output matrix is non-zero which implies that both eigenvalues in the jordan block are observable. However, the distinct eigenvalue is not observable, since the third element of the output matrix is zero. As a result, we observed that the first and the second state variables are observable, and the third state variable is not observable.

We could have determined all this information from the columns of the controllability and the rows of the observability matrices as well, but direct observation is obviously more efficient.

Designating the controllable, uncontrollable, observable, and unobservable portions by the subscripts c, $\bar{\text{c}}$, o, and $\bar{\text{o}}$, respectively, we have

$$x_{c,o} = x_1, \quad x_{c,\bar{o}} = x_3, \quad x_{\bar{c},o} = x_2, \text{ or } \quad \begin{bmatrix} x_{c,o} \\ x_{c,\bar{o}} \\ x_{\bar{c},o} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Therefore, the transformation matrix is

$$\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x_{c,o} \\ x_{c,\bar{o}} \\ x_{\bar{c},o} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{c,o} \\ x_{c,\bar{o}} \\ x_{\bar{c},o} \end{bmatrix}.$$

The system description after the transformation becomes

$$\begin{bmatrix} \dot{x}_{c,o}(t) \\ \dot{x}_{c,\bar{o}}(t) \\ \dot{x}_{\bar{c},o}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{c,o}(t) \\ x_{c,\bar{o}}(t) \\ x_{\bar{c},o}(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{c,o}(t) \\ x_{c,\bar{o}}(t) \\ x_{\bar{c},o}(t) \end{bmatrix}.$$

After some simplifications, we get the kalman decomposition.

$$\left[\begin{array}{c} \dot{x}_{c,o}(t) \\ \hline \dot{x}_{c,\bar{o}}(t) \\ \hline \dot{x}_{\bar{c},o}(t) \end{array}\right] = \left[\begin{array}{c|c|c} A_{c,o} & 0 & * \\ \hline * & A_{c,\bar{o}} & * \\ \hline 0 & 0 & A_{\bar{c},o} \end{array}\right] \left[\begin{array}{c} x_{c,o}(t) \\ \hline x_{c,\bar{o}}(t) \\ \hline x_{\bar{c},o}(t) \end{array}\right] + \left[\begin{array}{c} B_{c,o} \\ \hline B_{c,\bar{o}} \\ \hline 0 \end{array}\right] u(t),$$

$$= \left[\begin{array}{c|c|c} -1 & 0 & 1 \\ \hline 0 & -2 & 0 \\ \hline 0 & 0 & -1 \end{array}\right] \left[\begin{array}{c} x_{c,o}(t) \\ \hline x_{c,\bar{o}}(t) \\ \hline x_{\bar{c},o}(t) \end{array}\right] + \left[\begin{array}{c} 1 \\ \hline 1 \\ \hline 0 \end{array}\right] u(t),$$

$$y(t) = \left[\begin{array}{c|c|c} C_{c,o} & 0 & C_{\bar{c},o} \end{array}\right] \left[\begin{array}{c} x_{c,o}(t) \\ \hline x_{c,\bar{o}}(t) \\ \hline x_{\bar{c},o}(t) \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_{c,o}(t) \\ \hline x_{c,\bar{o}}(t) \\ \hline x_{\bar{c},o}(t) \end{array}\right].$$

4. The transfer matrix of a control system is given by

$$H(s) = \begin{bmatrix} \dfrac{s+2}{(s+1)(s+3)} & \dfrac{s+2}{(s+1)^2(s+3)} \\ \dfrac{1}{s(s+3)} & \dfrac{3s+7}{s(s+1)^2(s+3)} \\ \dfrac{2/3}{(s+1)(s+3)} & \dfrac{1}{s(s+1)(s+3)} \end{bmatrix}.$$

Obtain its left or right coprime factorization. Determine the order of a controllable and observable realization of the system.

Solution: To determine the type of coprime factorization, we need to we consider the dimension of the transfer matrix. In this case, $H(s)$ is 3×2. As a result, a left coprime factorization will yield $H(s) = D^{-1}_{L_{3\times3}}(s)N_{L_{3\times2}}(s)$, and a right coprime factorization will yield $H(s) = N_{R_{3\times2}}(s)D^{-1}_{R_{2\times2}}(s)$. Since a right coprime factorization will yield a smaller size $D(s)$ matrix, we prefer to obtain a right coprime factorization.

We first obtain an initial right factorization of $H = N_0 D_0^{-1}$, such that

$$H(s) = \begin{bmatrix} \dfrac{s+2}{(s+1)(s+3)} & \dfrac{s+2}{(s+1)^2(s+3)} \\ \dfrac{1}{s(s+3)} & \dfrac{3s+7}{s(s+1)^2(s+3)} \\ \dfrac{2/3}{(s+1)(s+3)} & \dfrac{1}{s(s+1)(s+3)} \end{bmatrix}$$

$$= \begin{bmatrix} s(s+1)(s+2) & s(s+2) \\ (s+1)^2 & 3s+7 \\ (2/3)s(s+1) & s+1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{s(s+1)^2(s+3)} & 0 \\ 0 & \dfrac{1}{s(s+1)^2(s+3)} \end{bmatrix}$$

$$= N_0(s) D_0^{-1}(s).$$

From the above factorization, we get

$$D_0 = \begin{bmatrix} s(s+1)^2(s+3) & 0 \\ 0 & s(s+1)^2(s+3) \end{bmatrix}.$$

Next, we form an augmented matrix from N_0 and D_0, perform column operations until we obtain the Hermite form, and factor out common polynomials from each column.

$$\begin{bmatrix} N_0(s) \\ \hline D_0(s) \end{bmatrix} = \begin{bmatrix} s(s+1)(s+2) & s(s+2) \\ (s+1)^2 & 3s+7 \\ (2/3)s(s+1) & s+1 \\ \hline s(s+1)^2(s+3) & 0 \\ 0 & s(s+1)^2(s+3) \end{bmatrix}.$$

Dividing the first column by $(s+1)$, we get

$$\begin{bmatrix} N_1(s) \\ \hline D_1(s) \end{bmatrix} = \begin{bmatrix} s(s+2) & s(s+2) \\ (s+1) & 3s+7 \\ (2/3)s & s+1 \\ \hline s(s+1)(s+3) & 0 \\ 0 & s(s+1)^2(s+3) \end{bmatrix}.$$

Subtracting the first column from the second, we get

$$\begin{bmatrix} N_2(s) \\ \hline D_2(s) \end{bmatrix} = \begin{bmatrix} s(s+2) & 0 \\ (s+1) & 2(s+3) \\ (2/3)s & (1/3)(s+3) \\ \hline s(s+1)(s+3) & -s(s+1)(s+3) \\ 0 & s(s+1)^2(s+3) \end{bmatrix}.$$

Dividing the second column by $(s+3)$, we get

$$\begin{bmatrix} N_3(s) \\ \hline D_3(s) \end{bmatrix} = \begin{bmatrix} s(s+2) & 0 \\ (s+1) & 2 \\ (2/3)s & 1/3 \\ \hline s(s+1)(s+3) & -s(s+1) \\ 0 & s(s+1)^2 \end{bmatrix}.$$

The last operation resulted in a coprime factorization, since the rank of the above augmented matrix will not drop for any value of s. (If $s \neq 0$, or $s \neq -2$; then the first two rows are linearly independent. If $s = 0$, or $s = -2$; then the second and the third rows are linearly independent.) As a result, one coprime factorization of the control system is

$$H(s) = N(s)D^{-1}(s) = \begin{bmatrix} s(s+2) & 0 \\ (s+1) & 2 \\ (2/3)s & 1/3 \end{bmatrix} \begin{bmatrix} s(s+1)(s+3) & -s(s+1) \\ 0 & s(s+1)^2 \end{bmatrix}^{-1}$$

To determine the order of the minimal or controllable and observable system, we determine $\det(D(s))$, which is a polynomial of order 6. Therefore, the order of a controllable and observable realization of the system is 6.