**EE 431** | **Exam#2** 75 minutes | **Dec. 07, 2004**



1. A continuous-time linear control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \boldsymbol{x}(t),$$

where $u$, $\boldsymbol{x}$, and $y$ are the input, the state, and the output variables, respectively.

(a) Design a control action for the system using a linear combination of the dirac-delta function and its derivatives as the input, such that the state $\boldsymbol{x}(0_+)$ is obtained from the initial state $\boldsymbol{x}(0_-)$, where

$$\boldsymbol{x}(0_-) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ and } \boldsymbol{x}(0_+) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Here, $t = 0_-$ represents $t = \lim_{\tau \nearrow 0}(\tau)$, and $t = 0_+$ represents $t = \lim_{\tau \searrow 0}(\tau)$. (25pts)

HINT: The $n$th derivative of the dirac-delta function (distribution) $\delta^{(n)}$ for a natural number $n$ is such that

$$\int_S f(\tau)\delta^{(n)}(\tau - t)\,\mathrm{d}\tau = (-1)^n f^{(n)}(t),$$

where $f$ is a $n$ times differentiable function, $\mathcal{S}$ is a set, and $t, \tau \in \mathcal{S}$.

(b) Design a continuous and integrable control action $u(t)$ for $0 \le t \le 1$, such that the state $\boldsymbol{x}(1)$ is obtained from the initial state $\boldsymbol{x}(0)$, where

$$\boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ and } \boldsymbol{x}(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(25pts)

HINT: For a natural number $n$ and a scalar $\alpha$,

$$\int t^n e^{\alpha t}\,\mathrm{d}t = \left(t^n - \frac{n}{\alpha}t^{n-1} + \frac{n(n-1)}{\alpha^2}t^{n-2} - \ldots + (-1)^n \frac{n!}{\alpha^n}\right)\frac{e^{\alpha t}}{\alpha}.$$

2. The transfer matrix of a control system is given by

$$H(s) = \begin{bmatrix} \dfrac{1}{s(s+1)} & \dfrac{1}{s} \\ \dfrac{s+2}{s^2(s+1)} & 0 \end{bmatrix}.$$

(a) Obtain its left or right coprime factorization. (25pts)

(b) Obtain a controllable and observable state-space realization of the system. (25pts)

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$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \boldsymbol{x}(t),$$

where $u$, $\boldsymbol{x}$, and $y$ are the input, the state, and the output variables, respectively.

(a) Design a control action for the system using a linear combination of the dirac-delta function and its derivatives as the input, such that the state $\boldsymbol{x}(0_+)$ is obtained from the initial state $\boldsymbol{x}(0_-)$, where

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Here, $t = 0_-$ represents $t = \lim_{\tau \nearrow 0}(\tau)$, and $t = 0_+$ represents $t = \lim_{\tau \searrow 0}(\tau)$.

HINT: The $n$th derivative of the dirac-delta function (distribution) $\delta^{(n)}$ for a natural number $n$ is such that

$$\int_{\mathcal{S}} f(\tau)\delta^{(n)}(\tau - t)\,\mathrm{d}\tau = (-1)^n f^{(n)}(t),$$

where $f$ is a $n$ times differentiable function, $\mathcal{S}$ is a set, and $t, \tau \in \mathcal{S}$.

**Solution:** The general solution to the state-space representation of a linear system described by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

is obtained from

$$\boldsymbol{x}(t) = \Phi(t,0)\boldsymbol{x}(0) + \int_0^t \Phi(t,\tau)B\boldsymbol{u}(\tau)\,\mathrm{d}\tau,$$

where $\boldsymbol{u}$ and $\boldsymbol{x}$ are the input and the state variables, respectively, and

$$\Phi(t, t_0) = e^{A(t-t_0)}.$$

To instantaneously change the states, we need to apply a linear combination of the dirac-delta function and its derivatives as the input. We also know that if the system is reachable, there exists such a linear combination. Since the system is second order, we only need two terms in this linear combination. So, we let

$$u(t) = u_0\delta(t) + u_1\dot{\delta}(t).$$

Substituting the input to the solution equation for the state variable, we get

$$\begin{aligned}
\boldsymbol{x}(t) &= e^{At}\boldsymbol{x}(0_-) + \int_{0_-}^t e^{A(t-\tau)}B\left(u_0\delta(\tau) + u_1\dot{\delta}(\tau)\right)\mathrm{d}\tau \\
&= e^{At}\boldsymbol{x}(0_-) + \left(e^{A(t-\tau)}Bu_0 + (-1)\frac{\mathrm{d}}{\mathrm{d}\tau}\left(e^{A(t-\tau)}Bu_1\right)\right)_{\tau=0} \\
&= e^{At}\boldsymbol{x}(0_-) + \left(e^{At}Bu_0 + e^{At}ABu_1\right) \\
&= e^{At}\boldsymbol{x}(0_-) + e^{At}\begin{bmatrix} B & AB \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix};
\end{aligned}$$

or

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \mathcal{C}(A,B)^{-1}\left(e^{-At}\boldsymbol{x}(t) - \boldsymbol{x}(0_-)\right),$$

where $\mathcal{C}(A,B)$ is the controllability matrix. In our case, the system order $n = 2$, and

$$\mathcal{C}(A,B) = \left[\ B \mid AB \mid \cdots \mid A^{n-1}B\ \right] = \left[\ B \mid AB\ \right] = \left[\begin{array}{c|c} 0 & 1 \\ 1 & -1 \end{array}\right].$$

For $t = 0_+$, we get

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$u(t) = \dot{\delta}(t).$$

(b) Design a continuous and integrable control action $u(t)$ for $0 \le t \le 1$, such that the state $\boldsymbol{x}(1)$ is obtained from the initial state $\boldsymbol{x}(0)$, where

$$\boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ and } \boldsymbol{x}(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

HINT: For a natural number $n$ and a scalar $\alpha$,

$$\int t^n e^{\alpha t}\, dt = \left(t^n - \frac{n}{\alpha}t^{n-1} + \frac{n(n-1)}{\alpha^2}t^{n-2} - \ldots + (-1)^n\frac{n!}{\alpha^n}\right)\frac{e^{\alpha t}}{\alpha}.$$

**Solution:** The general solution to the state-space representation of a linear system described by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

is obtained from

$$\boldsymbol{x}(t) = \Phi(t,0)\boldsymbol{x}(0) + \int_0^t \Phi(t,\tau)B\boldsymbol{u}(\tau)\, d\tau,$$

where $\boldsymbol{u}$ and $\boldsymbol{x}$ are the input and the state variables, respectively, and

$$\Phi(t,t_0) = e^{A(t-t_0)}.$$

In our case, the state matrix $A$ is in jordan form, and

$$\Phi(t,t_0) = e^{A(t-t_0)} = \begin{bmatrix} e^{-(t-t_0)} & (t-t_0)e^{-(t-t_0)} \\ 0 & e^{-(t-t_0)} \end{bmatrix}.$$

One method to solve the given control problem is to separate the control signal into two portions, such that one of the portions can be moved out of the integral, and the integral term is invertible. So, we let

$$u(\tau) = \left(e^{A(t-\tau)}B\right)^T \boldsymbol{\xi}(t),$$

where $\boldsymbol{\xi}$ needs to be determined from the given conditions. With this control signal, we get

$$\begin{aligned}\boldsymbol{x}(t) &= e^{At}\boldsymbol{x}(0) + \left(\int_0^t e^{A(t-\tau)}BB^Te^{A(t-\tau)^T}\,\mathrm{d}\tau\right)\boldsymbol{\xi}(t)\\ &= e^{At}\boldsymbol{x}(0) + \mathfrak{C}(0,t)\boldsymbol{\xi}(t),\end{aligned}$$

where the controllability grammian

$$\mathfrak{C}(0,t) = \int_0^t e^{A(t-\tau)}BB^Te^{A(t-\tau)^T}\,\mathrm{d}\tau.$$

Since the controllability grammian always has an inverse, if the system is reachable; we get

$$\boldsymbol{\xi}(t) = \mathfrak{C}^{-1}(0,t)\left(\boldsymbol{x}(t) - e^{At}\boldsymbol{x}(0)\right),$$

or

$$u(\tau) = \left(e^{A(t-\tau)}B\right)^T\boldsymbol{\xi}(t) = \left(e^{A(t-\tau)}B\right)^T\mathfrak{C}^{-1}(0,t)\left(\boldsymbol{x}(t) - e^{At}\boldsymbol{x}(0)\right).$$

To determine the control signal, we first compute

$$e^{A(t-\tau)}B = \begin{bmatrix} e^{-(t-\tau)} & (t-\tau)e^{-(t-\tau)} \\ 0 & e^{-(t-\tau)}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}(t-\tau)e^{-(t-\tau)}\\ e^{-(t-\tau)}\end{bmatrix},$$

and

$$\begin{aligned}\mathfrak{C}(0,t) &= \int_0^t e^{A(t-\tau)}B\left(e^{A(t-\tau)}B\right)^T\mathrm{d}\tau\\ &= \int_0^t \begin{bmatrix}(t-\tau)^2e^{-2(t-\tau)} & (t-\tau)e^{-2(t-\tau)}\\ (t-\tau)e^{-2(t-\tau)} & e^{-2(t-\tau)}\end{bmatrix}\mathrm{d}\tau\\ &= \int_0^t \begin{bmatrix}\mu^2e^{-2\mu} & \mu e^{-2\mu}\\ \mu e^{-2\mu} & e^{-2\mu}\end{bmatrix}\mathrm{d}\mu,\end{aligned}$$

where $\mu = t - \tau$. Using the indefinite integral

$$\int t^ne^{\alpha t}\,\mathrm{d}t = \left(t^n - \frac{n}{\alpha}t^{n-1} + \frac{n(n-1)}{\alpha^2}t^{n-2} - \ldots + (-1)^n\frac{n!}{\alpha^n}\right)\frac{e^{\alpha t}}{\alpha},$$

we get

$$\begin{aligned}\mathfrak{C}(0,t) &= \begin{bmatrix}(\mu^2 - (2/(-2))\mu + (2/(-2)^2))e^{-2\mu}/(-2) & (\mu - (1/(-2)))e^{-2\mu}/(-2)\\ (\mu - (1/(-2)))e^{-2\mu}/(-2) & e^{-2\mu}/(-2)\end{bmatrix}_{\mu=0}^{\mu=t}\\ &= \begin{bmatrix}-(\mu^2+\mu+1/2)e^{-2\mu}/2 & -(\mu+1/2)e^{-2\mu}/2\\ -(\mu+1/2)e^{-2\mu}/2 & -e^{-2\mu}/2\end{bmatrix}_{\mu=0}^{\mu=t}\\ &= \begin{bmatrix}(-(t^2+t+1/2)e^{-2t}/2) - (-(1/2)/2) & (-(t+1/2)e^{-2t}/2) - (-(1/2)/2)\\ (-(t+1/2)e^{-2t}/2) - (-(1/2)/2) & (-e^{-2t}/2) - (-1/2)\end{bmatrix}\\ &= \begin{bmatrix}-(t^2/2+t/2+1/4)e^{-2t} + 1/4 & -(t/2+1/4)e^{-2t} + 1/4\\ -(t/2+1/4)e^{-2t} + 1/4 & -(1/2)e^{-2t} + 1/2\end{bmatrix}\end{aligned}$$

In our case, we need

$$\mathfrak{C}(0,1) = \begin{bmatrix} (1-5e^{-2})/4 & (1-3e^{-2})/4 \\ (1-3e^{-2})/4 & (1-e^{-2})/2 \end{bmatrix} = \begin{bmatrix} 0.0808 & 0.1485 \\ 0.1485 & 0.4323 \end{bmatrix},$$

and

$$\mathfrak{C}^{-1}(0,1) = \begin{bmatrix} 33.5297 & -11.5169 \\ -11.5169 & 6.2689 \end{bmatrix}.$$

As a result,

$$\begin{aligned} u(\tau) &= \left(e^{A(1-\tau)}B\right)^T \mathfrak{C}^{-1}(0,1)\left(\boldsymbol{x}(1) - e^{At}\boldsymbol{x}(0)\right) \\ &= \begin{bmatrix} (1-\tau)e^{-(1-\tau)} \\ e^{-(1-\tau)} \end{bmatrix}^T \begin{bmatrix} 33.5297 & -11.5169 \\ -11.5169 & 6.2689 \end{bmatrix} \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{At} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \\ &= 45.0466(1-\tau)e^{-(1-\tau)} - 17.7858e^{-(1-\tau)} \\ &= 10.0287e^{\tau} - 16.5717\tau e^{\tau}, \end{aligned}$$

or

$$u(t) = 10.0287e^{t} - 16.5717te^{t} \text{ for } 0 \le t \le 1.$$

2. The transfer matrix of a control system is given by

$$H(s) = \begin{bmatrix} \dfrac{1}{s(s+1)} & \dfrac{1}{s} \\ \dfrac{s+2}{s^2(s+1)} & 0 \end{bmatrix}.$$

(a) Obtain its left or right coprime factorization.

**Solution:**

**Right coprime factorization:** For the right coprime factorization, we start with an initial right factorization of $H = N_0 D_0^{-1}$, such that

$$\begin{aligned} H(s) &= \begin{bmatrix} \dfrac{1}{s(s+1)} & \dfrac{1}{s} \\ \dfrac{s+2}{s^2(s+1)} & 0 \end{bmatrix} \\ &= \begin{bmatrix} s & s(s+1) \\ s+2 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{1}{s^2(s+1)} & 0 \\ 0 & \dfrac{1}{s^2(s+1)} \end{bmatrix} \\ &= N_0(s) D_0^{-1}(s). \end{aligned}$$

From the above factorization, we get

$$D_0 = \begin{bmatrix} s^2(s+1) & 0 \\ 0 & s^2(s+1) \end{bmatrix}.$$

Next, we form an augmented matrix from $N_0$ and $D_0$, perform column operations until we obtain the Hermite form, and factor out common polynomials from each column.

$$\left[\begin{array}{c} N_0(s) \\ \hline D_0(s) \end{array}\right] = \left[\begin{array}{cc} s & s(s+1) \\ s+2 & 0 \\ \hline s^2(s+1) & 0 \\ 0 & s^2(s+1) \end{array}\right].$$

Dividing the second column by $s(s+1)$, we get

$$\left[\begin{array}{c} N_1(s) \\ \hline D_1(s) \end{array}\right] = \left[\begin{array}{cc} s & 1 \\ s+2 & 0 \\ \hline s^2(s+1) & 0 \\ 0 & s \end{array}\right].$$

Interchanging the first and second columns, we get

$$\left[\begin{array}{c} N_2(s) \\ \hline D_2(s) \end{array}\right] = \left[\begin{array}{cc} 1 & s \\ 0 & s+2 \\ \hline 0 & s^2(s+1) \\ s & 0 \end{array}\right].$$

Multiplying the first column by $-s$ and adding to the second, we get

$$\left[\begin{array}{c} N_3(s) \\ \hline D_3(s) \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & s+2 \\ \hline 0 & s^2(s+1) \\ s & -s^2 \end{array}\right].$$

The last operation resulted in a coprime factorization, since the rank of the above augmented matrix will not drop for any value of $s$, and the degree of the determinant of $D(s)$ is the same as the sum of its highest column degrees. As a result, one right coprime factorization of the control system is

$$H(s) = N(s)D^{-1}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 0 & s^2(s+1) \\ s & -s^2 \end{bmatrix}^{-1}.$$

**Left coprime factorization:** For the left coprime factorization, we start with an initial left factorization of $H = D_0^{-1}N_0$, such that

$$H(s) = \begin{bmatrix} \frac{1}{s(s+1)} & \frac{1}{s} \\ \frac{s+2}{s^2(s+1)} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s^2(s+1)} & 0 \\ 0 & \frac{1}{s^2(s+1)} \end{bmatrix} \begin{bmatrix} s & s(s+1) \\ s+2 & 0 \end{bmatrix}$$

$$= D_0^{-1}(s)N_0(s).$$

From the above factorization, we get

$$D_0 = \begin{bmatrix} s^2(s+1) & 0 \\ 0 & s^2(s+1) \end{bmatrix}.$$

Next, we form an augmented matrix from $N_0$ and $D_0$, and perform row operations until we obtain the Hermite form.

$$\left[\, N_0(s) \,\middle|\, D_0(s) \,\right] = \left[\begin{array}{cc|cc} s & s(s+1) & s^2(s+1) & 0 \\ s+2 & 0 & 0 & s^2(s+1) \end{array}\right].$$

Dividing the first row by $s$, we get

$$\left[\, N_1(s) \,\middle|\, D_1(s) \,\right] = \left[\begin{array}{cc|cc} 1 & s+1 & s(s+1) & 0 \\ s+2 & 0 & 0 & s^2(s+1) \end{array}\right].$$

Multiplying the first row by $-(s+1)$ and adding to the second, we get

$$\left[\, N_2(s) \,\middle|\, D_2(s) \,\right] = \left[\begin{array}{cc|cc} 1 & s+1 & s(s+1) & 0 \\ 0 & -(s+1)(s+2) & -s(s+1)(s+2) & s^2(s+1) \end{array}\right].$$

Dividing the second row by $-(s+1)$, we get

$$\left[\, N_3(s) \,\middle|\, D_3(s) \,\right] = \left[\begin{array}{cc|cc} 1 & s+1 & s(s+1) & 0 \\ 0 & s+2 & s(s+2) & -s^2 \end{array}\right].$$

The last operation resulted in a coprime factorization, since the rank of the above augmented matrix will not drop for any value of $s$, and the degree of the determinant of $D(s)$ is the same as the sum of its highest row degrees. As a result, one left coprime factorization of the control system is

$$H(s) = D^{-1}(s)N(s) = \begin{bmatrix} s(s+1) & 0 \\ s(s+2) & -s^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & s+1 \\ 0 & s+2 \end{bmatrix}.$$

(b) Obtain a controllable and observable state-space realization of the system.

**Solution:** A controllable and observable state-space realization is a minimal realization. We may use a right or a left coprime factorization, and in each case, we will get a controllable and observable system.

**Using right coprime factorization:** In this case, we start with a right coprime factorization of the system, and for the given transfer matrix, we have

$$H(s) = N(s)D^{-1}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 0 & s^2(s+1) \\ s & -s^2 \end{bmatrix}^{-1}.$$

Next, we need to decompose $D(s)$ and $N(s)$, such that

$$D(s) = D_{h_c} S_c(s) + D_{l_c} \Psi_c(s),$$

and

$$N(s) = N_{l_c} \Psi_c(s),$$

where

$$S_c(s) = \begin{bmatrix} s^{k_1} & & \\ & s^{k_2} & \\ & & \ddots \end{bmatrix} \text{ and } \Psi_c(s) = \begin{bmatrix} \begin{matrix} 1 \\ \vdots \\ s^{k_1-1} \end{matrix} & & \\ & \begin{matrix} 1 \\ \vdots \\ s^{k_2-1} \end{matrix} & \\ & & \ddots \end{bmatrix}$$

are block-diagonal matrices, and $k_i$ is the highest degree of the polynomials on the $i$th column of $D(s)$. In our case, $k_1 = 1$, $k_2 = 3$,

$$S_c(s) = \left[\begin{array}{c|c} s & 0 \\ \hline 0 & s^3 \end{array}\right], \text{ and } \Psi_c(s) = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \\ 0 & s \\ 0 & s^2 \end{array}\right].$$

The decompositions become

$$D(s) = \begin{bmatrix} 0 & s^2(s+1) \\ s & -s^2 \end{bmatrix} = \begin{bmatrix} 0 & s^3+s^2 \\ s & -s^2 \end{bmatrix}$$

$$= D_{h_c}S_c(s) + D_{l_c}\Psi_c(s) = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right]\left[\begin{array}{c|c} s & 0 \\ \hline 0 & s^3 \end{array}\right] + \left[\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 \end{array}\right]\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \\ 0 & s \\ 0 & s^2 \end{array}\right],$$

and

$$N(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+2 \end{bmatrix} = N_{l_c}\Psi_c(s) = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 2 & 1 & 0 \end{array}\right]\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \\ 0 & s \\ 0 & s^2 \end{array}\right].$$

The controller canonical-form realization is, then, given by

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= (A_c^0 - B_c^0 D_{h_c}^{-1} D_{l_c} C_c^0)\boldsymbol{x}(t) + (B_c^0 D_{h_c}^{-1})\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= (N_{l_c} C_c^0)\boldsymbol{x}(t), \end{aligned}$$

where

$$A_c^0 = \left[\begin{array}{cccc|cccc|c} 0 & 1 & \cdots & 0 & & & & & \\ \vdots & \ddots & \ddots & \vdots & & & & & \\ 0 & 0 & \cdots & 1 & & & & & \\ 0 & 0 & \cdots & 0 & & & & & \\ \hline & & & & 0 & 1 & \cdots & 0 & \\ & & & & \vdots & \ddots & \ddots & \vdots & \\ & & & & 0 & 0 & \cdots & 1 & \\ & & & & 0 & 0 & \cdots & 0 & \\ \hline & & & & & & & & \ddots \end{array}\right], \qquad B_c^0 = \left[\begin{array}{c|c|c} 0 & & \\ \vdots & & \\ 0 & & \\ 1 & & \\ \hline & 0 & \\ & \vdots & \\ & 0 & \\ & 1 & \\ \hline & & \ddots \end{array}\right],$$

and $C_c^0$ is the identity matrix with dimension $\sum_i k_i$. In our case,

$$A_c^0 = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \qquad B_c^0 = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right],$$

and $C_c^0$ is the 4 dimensional identity matrix. So,

$$A_c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$B_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and

$$C_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}.$$

Therefore, one possible state-space representation of the system is given by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \boldsymbol{u}(t),$$

$$\boldsymbol{y}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \boldsymbol{x}(t),$$

where $\boldsymbol{u}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ are the input, the state, and the output variables, respectively.

**Using left coprime factorization:** In this case, we start with a left coprime factorization of the system, and for the given transfer matrix, we have

$$H(s) = D^{-1}(s)N(s) = \begin{bmatrix} s(s+1) & 0 \\ s(s+2) & -s^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & s+1 \\ 0 & s+2 \end{bmatrix}.$$

Next, we need to decompose $D(s)$ and $N(s)$, such that

$$D(s) = S_r(s)D_{h_r} + \Psi_r(s)D_{l_r},$$

and

$$N(s) = \Psi_r(s)N_{l_r},$$

where

$$S_r(s) = \begin{bmatrix} s^{l_1} & & \\ & s^{l_2} & \\ & & \ddots \end{bmatrix} \text{ and } \Psi_r(s) = \begin{bmatrix} s^{l_1-1} \ \cdots \ 1 & & \\ & s^{l_2-1} \ \cdots \ 1 & \\ & & \ddots \end{bmatrix}$$

are block-diagonal matrices, and $l_i$ is the highest degree of the polynomials on the $i$th row of $D(s)$. In our case, $l_1 = 2$, $l_2 = 2$,

$$S_r(s) = \left[\begin{array}{c|c} s^2 & 0 \\ \hline 0 & s^2 \end{array}\right], \text{ and } \Psi_r(s) = \left[\begin{array}{cc|cc} s & 1 & 0 & 0 \\ \hline 0 & 0 & s & 1 \end{array}\right].$$

The decompositions become

$$D(s) = \begin{bmatrix} s(s+1) & 0 \\ s(s+2) & -s^2 \end{bmatrix} = \begin{bmatrix} s^2+s & 0 \\ s^2+2s & -s^2 \end{bmatrix}$$

$$= S_r(s)D_{h_r} + \Psi_r(s)D_{l_r} = \left[\begin{array}{c|c} s^2 & 0 \\ \hline 0 & s^2 \end{array}\right]\left[\begin{array}{c|c} 1 & 0 \\ \hline 1 & -1 \end{array}\right] + \left[\begin{array}{cc|cc} s & 1 & 0 & 0 \\ \hline 0 & 0 & s & 1 \end{array}\right]\left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \\ \hline 2 & 0 \\ 0 & 0 \end{array}\right],$$

and

$$N(s) = \begin{bmatrix} 1 & s+1 \\ 0 & s+2 \end{bmatrix} = \Psi_r(s)N_{l_r} = \left[\begin{array}{cc|cc} s & 1 & 0 & 0 \\ \hline 0 & 0 & s & 1 \end{array}\right]\left[\begin{array}{c|c} 0 & 1 \\ 1 & 1 \\ \hline 0 & 1 \\ 0 & 2 \end{array}\right].$$

The controller canonical-form realization is, then, given by

$$\dot{\boldsymbol{x}}(t) = \left(A_o^0 - B_o^0 D_{l_r} D_{h_r}^{-1} C_o^0\right)\boldsymbol{x}(t) + \left(B_o^0 N_{l_r}\right)\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = \left(D_{h_r}^{-1} C_o^0\right)\boldsymbol{x}(t),$$

where

$$A_o^0 = \left[\begin{array}{c|c|c} \begin{matrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{matrix} & & \\ \hline & \begin{matrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{matrix} & \\ \hline & & \ddots \end{array}\right],$$

$B_o^0$ is the identity matrix with dimension $\sum_i l_i$, and

$$C_o^0 = \left[\begin{array}{c|c|c} 1 \; 0 \; \cdots \; 0 & & \\ \hline & 1 \; 0 \; \cdots \; 0 & \\ \hline & & \ddots \end{array}\right].$$

In our case,

$$A_o^0 = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$B_o^0$ is the 4 dimensional identity matrix, and

$$C_o^0 = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right].$$

So,

$$A_o = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix},$$

and

$$C_c = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

Therefore, another possible state-space representation of the system is given by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \boldsymbol{u}(t),$$

$$\boldsymbol{y}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \boldsymbol{x}(t),$$

where $\boldsymbol{u}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ are the input, the state, and the output variables, respectively.