EE 431 | Exam#2 | Dec. 04, 2001

75 minutes

1. A control system is described in state-space representation, such that

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t),$$

where $\mathbf{u}$, $\mathbf{x}$, and $\mathbf{y}$ are the input, the state, and the output variables, respectively. For the following A, B, C, and D matrices, determine whether the system is asymptotically stable, marginally stable, or unstable; and whether it is bounded-input-bounded-output stable or not. Justify your answer.

(a)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \text{ and } D = 0.$$

(10pts)

(b)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \text{ and } D = 0.$$

(10pts)

(c)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \text{ and } D = 0.$$

(10pts)

2. A discrete-time linear control system is described by

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.3 & 1.0 \\ -0.2 & 0.5 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.50 \\ 0.25 \end{bmatrix} u(k),$$
$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(k),$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Determine whether or not an initial condition can be uniquely determined by observing the future values of the output and control. If such an observation is possible, then determine the initial condition $\mathbf{x}(0)$, when the output sequence is $\{\, y(k) \mid k = 0, 1, 2, \dots \} = \{\, 1,\ 3.65,\ 1.645,\ -0.5365, \dots \}$ for the input sequence $\{\, u(k) \mid k = 0, 1, 2, \dots \} = \{\, 1,\ -1,\ -1,\ 1, \dots \}$. (25pts)

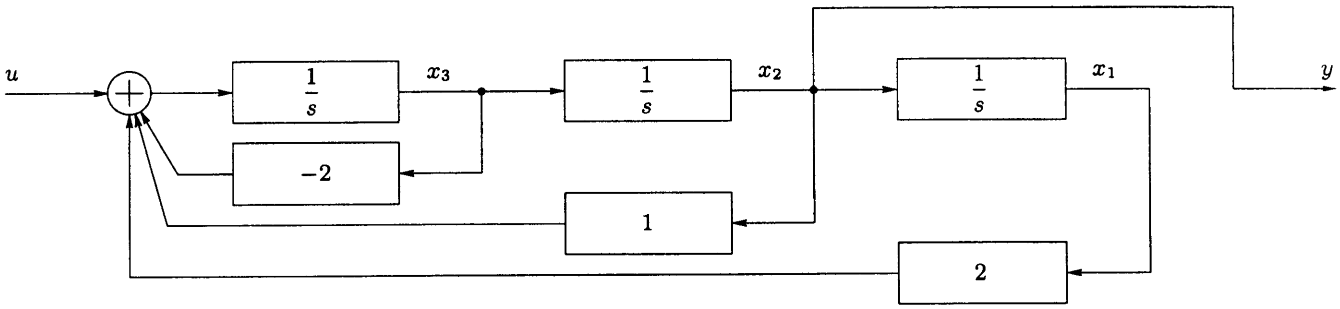
3. During the design of the following control system, some of the locations of the sensors and actuators need to be chosen such that the resultant system is minimal. Determine all the possible values of the constants a, b, c, d, and e that would result in a minimal system.

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & b \\ a & 0 \\ -1 & c \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & d & e \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{u}$, $\mathbf{x}$, and $\mathbf{y}$ are the input, the state, and the output variables, respectively. (20pts)

4. The block diagram of a control system is given below.



Design a proportional-integral-derivative (PID) controller

$$u(t) = K_P y(t) + K_I \int^t y(\tau)\,\mathrm{d}\tau + K_D \dot{y}(t),$$

where u and y are the input and the output variables, respectively, such that the closed-loop poles of the system are at $s = -1$ and $s = -2 \pm j1$. Here, K_P, K_I, and K_D are the proportional, the integral, and the derivative constants, respectively. Assume that the internal states x_1, x_2, and x_3 are available to the controller. (25pts)

1. A control system is described in state-space representation, such that

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t),$$

where $\mathbf{u}$, $\mathbf{x}$, and $\mathbf{y}$ are the input, the state, and the output variables, respectively. For the following A, B, C, and D matrices, determine whether the system is asymptotically stable, marginally stable, or unstable; and whether it is bounded-input-bounded-output stable or not. Justify your answer.

(a)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \text{ and } D = 0.$$

Solution: In order to determine the stability of the system, we first need to determine its eigenvalues or poles. Since in this case, the state matrix A is diagonal, we observe the eigenvalues directly from the diagonal elements as $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = -1$. The eigenvalue λ_3 has a negative real part, and it would generate an asymptotically stable response. The eigenvalues λ_1 and λ_2 are both zero, and each would generate a constant response individually. However, if they are cascaded, a constant response generated by the first one would result in a ramp response by the second one. In this case, since the state matrix A is diagonal, the two zero-valued eigenvalues don't effect each other, or they are not cascaded. Therefore, each of the eigenvalues λ_1 and λ_2 would generate a constant response resulting in a marginally stable response. Since there are no more eigenvalues, we conclude that the system is marginally stable.

In order to determine the bounded-input-bounded-output stability of the system, we may determine the transfer function and observe the poles of the system after all the reductions. The transfer function of the system is given by

$$\mathcal{L}\left[\mathbf{y}\right](s) = \left(C(sI - A)^{-1}B + D\right)\mathcal{L}\left[\mathbf{u}\right](s)$$

where $\mathcal{L}\left[(\cdot)\right](s)$ is the Laplace transform, and I is the appropriately dimensioned identity matrix. In our case,

$$C(sI - A)^{-1}B + D = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/s & 0 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1/(s+1) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/s \\ 1/s \\ 1/(s+1) \end{bmatrix} = \frac{1}{s} + \frac{1}{s} + \frac{1}{s+1} = \frac{3s+2}{s(s+1)}.$$

We realize that only two poles of the system are visible in the transfer function; and one of them has a negative real part, and the other one is on the imaginary axis. Since the one on the imaginary axis would generate a ramp response for a step input, the system is not bounded-input-bounded-output stable.

In summary, the system is marginally stable, and it is not bounded-input-bounded-output stable.

(b)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \text{ and } D = 0.$$

Solution: Since the state matrix A is upper diagonal, we observe the eigenvalues directly from the diagonal elements as $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = -1$. The eigenvalue λ_3 has a negative real part, and it would generate an asymptotically stable response. The eigenvalues λ_1 and λ_2 are both zero, and each would generate a constant response individually. However, if they are cascaded, a constant response generated by the first one would result in a ramp response by the second one. In this case, since the state matrix A is in Jordan form, the two zero-valued eigenvalues are cascaded, and they effect each other. Therefore, the state corresponding to λ_2 would generate a constant response, and the state corresponding to λ_1 would then generate a ramp response. As a result, we conclude that the system is unstable.

In order to determine the bounded-input-bounded-output stability of the system, we may determine the transfer function and observe the poles of the system after all the reductions. The transfer function of the system is given by

$$\mathcal{L}\left[\,\mathbf{y}\,\right](s) = \left(C(sI - A)^{-1}B + D\right)\mathcal{L}\left[\,\mathbf{u}\,\right](s)$$

where $\mathcal{L}\left[\,(\cdot)\,\right](s)$ is the Laplace transform, and I is the appropriately dimensioned identity matrix. One method to determine the inverse of $(sI - A)$ is to use row operations on the augmented matrix $\left[\,(sI - A) \quad I\,\right]$ to generate $\left[\,I \quad (sI - A)^{-1}\,\right]$.

$$\left[\begin{array}{ccc|ccc} s & -1 & 0 & 1 & 0 & 0 \\ 0 & s & 0 & 0 & 1 & 0 \\ 0 & 0 & s+1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} s & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/s & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/(s+1) \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} s & 0 & 0 & 1 & 1/s & 0 \\ 0 & 1 & 0 & 0 & 1/s & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/(s+1) \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/s & 1/s^2 & 0 \\ 0 & 1 & 0 & 0 & 1/s & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/(s+1) \end{array}\right].$$

Therefore,

$$C(sI-A)^{-1}B+D = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/s & 1/s^2 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1/(s+1) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/s + 1/s^2 \\ 1/s \\ 1/(s+1) \end{bmatrix} = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s} + \frac{1}{s+1} = \frac{3s^2+3s+2}{s^2(s+1)}.$$

Because of the repeated pole at zero, the impulse response would contain a ramp function; and as a result the system is not bounded-input-bounded-output stable.

In summary, the system is unstable, and it is not bounded-input-bounded-output stable.

(c)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \text{ and } D = 0.$$

Solution: Since the state matrix A is upper diagonal, we observe the eigenvalues directly from the diagonal elements as $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = -1$. The eigenvalue λ_3 has a negative real part, and it would generate an asymptotically stable response. The eigenvalues λ_1 and λ_2 are both zero, and each would generate a constant response individually. However, if they are cascaded, a constant response generated by the first one would result in a ramp response by the second one. In this case, since the state matrix A is in Jordan form, the two zero-valued eigenvalues are cascaded, and they effect each other. Therefore, the state corresponding to λ_2 would generate a constant response, and the state corresponding to λ_1 would then generate a ramp response. As a result, we conclude that the system is unstable.

In order to determine the bounded-input-bounded-output stability of the system, we may determine the transfer function and observe the poles of the system after all the reductions. The transfer function of the system is given by

$$\mathcal{L}\left[\,\mathbf{y}\,\right](s) = \left(C(sI-A)^{-1}B + D\right)\mathcal{L}\left[\,\mathbf{u}\,\right](s)$$

where $\mathcal{L}\left[\,(\cdot)\,\right](s)$ is the Laplace transform, and I is the appropriately dimensioned identity matrix. One method to determine the inverse of $(sI - A)$ is to use row operations on the augmented matrix $\left[\ (sI-A)\quad I\ \right]$ to generate $\left[\ I \quad (sI-A)^{-1}\ \right]$.

$$\left[\begin{array}{ccc|ccc} s & -1 & 0 & 1 & 0 & 0 \\ 0 & s & 0 & 0 & 1 & 0 \\ 0 & 0 & s+1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} s & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/s & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/(s+1) \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} s & 0 & 0 & 1 & 1/s & 0 \\ 0 & 1 & 0 & 0 & 1/s & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/(s+1) \end{array}\right]$$

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Therefore,

$$C(sI-A)^{-1}B + D = \left[\ 1 \quad 1 \quad 1\ \right] \begin{bmatrix} 1/s & 1/s^2 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1/(s+1) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0$$

$$= \left[\ 1 \quad 1 \quad 1\ \right] \begin{bmatrix} 1/s \\ 0 \\ 1/(s+1) \end{bmatrix} = \frac{1}{s} + \frac{1}{s+1} = \frac{2s+1}{s(s+1)}.$$

We realize that only two poles of the system are visible in the transfer function; and one of them has a negative real part, and the other one is on the imaginary axis. Since the one on the imaginary axis would generate a ramp response for a step input, the system is not bounded-input-bounded-output stable.

In summary, the system is unstable, and it is not bounded-input-bounded-output stable.

2. A discrete-time linear control system is described by

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.3 & 1.0 \\ -0.2 & 0.5 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.50 \\ 0.25 \end{bmatrix} u(k),$$
$$y(k) = \left[\ 1 \quad 1\ \right] \mathbf{x}(k),$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Determine whether or not an initial condition can be uniquely determined by observing the future values of the output and control. If such an observation is possible, then determine the initial condition $\mathbf{x}(0)$, when the output sequence is $\{\, y(k) \mid k = 0,1,2,\ldots \,\} = \{\, 1,\ 3.65,\ 1.645,\ -0.5365, \ldots \,\}$ for the input sequence $\{\, u(k) \mid k = 0,1,2,\ldots \,\} = \{\, 1,\ -1,\ -1,\ 1, \ldots \,\}$.

Solution: The property of being able to uniquely determine the initial condition $\mathbf{x}(0)$ by observing the future values of the output and control is called the observability property. This property may be checked in a number of ways, where one such way is from the rank of the observability matrix

$$\mathcal{O}(C,A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Here, the nth order system is described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t),$$

where $\mathbf{u}$, $\mathbf{x}$, and $\mathbf{y}$ are the input, the state, and the output variables, respectively. In our case, the system order $n = 2$, and

$$\mathcal{O}(C,A) = \left[\begin{array}{c} C \\ \hline CA \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ \hline 0.1 & 1.5 \end{array}\right].$$

Since the rank of the observability matrix

$$\text{rank}[\,\mathcal{O}\,] = \begin{bmatrix} 1 & 1 \\ 0.1 & 1.5 \end{bmatrix} = 2,$$

which is the system order; the system is observable, and the initial condition can be uniquely determined by observing the future values of the output and control. To determine the initial condition $\mathbf{x}(0)$, we need at most two of the output values, since the order of the system is two. From the system equations,

$$y(0) = C\mathbf{x}(0)$$
$$y(1) = C\mathbf{x}(1) = CA\mathbf{x}(0) + CBu(0),$$

or in vector form

$$\begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix}\mathbf{x}(0) + \begin{bmatrix} 0 \\ CB \end{bmatrix}u(0).$$

We can solve for $\mathbf{x}(0)$ from the above equation, since the observability matrix is invertible.

$$\begin{aligned}
\mathbf{x}(0) &= \begin{bmatrix} C \\ CA \end{bmatrix}^{-1}\left(\begin{bmatrix} y(0) \\ y(1) \end{bmatrix} - \begin{bmatrix} 0 \\ CB \end{bmatrix}u(0)\right) \\
&= \begin{bmatrix} 1 & 1 \\ 0.1 & 1.5 \end{bmatrix}^{-1}\left(\begin{bmatrix} 1 \\ 3.65 \end{bmatrix} - \begin{bmatrix} 0 \\ 0.75 \end{bmatrix}1\right) \\
&= \frac{1}{1.4}\begin{bmatrix} 1.5 & -1 \\ -0.1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2.9 \end{bmatrix} = \frac{1}{1.4}\begin{bmatrix} -1.4 \\ 2.8 \end{bmatrix}.
\end{aligned}$$

Therefore, the initial condition

$$\mathbf{x}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

3. During the design of the following control system, some of the locations of the sensors and actuators need to be chosen such that the resultant system is minimal. Determine all the possible values of the constants a, b, c, d, and e that would result in a minimal system.

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & b \\ a & 0 \\ -1 & c \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & d & e \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{u}$, $\mathbf{x}$, and $\mathbf{y}$ are the input, the state, and the output variables, respectively.

Solution: For a minimal system, the system needs to be observable and reachable.

Observability: One method to check the observability of the system is by checking the rank of the observability matrix

$$\mathcal{O}(C,A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Here, the nth order system is described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t),$$

where $\mathbf{u}$, $\mathbf{x}$, and $\mathbf{y}$ are the input, the state, and the output variables, respectively. In our case, the system order $n = 3$, and

$$\mathcal{O}(C,A) = \begin{bmatrix} C \\ \hline CA \\ \hline CA^2 \end{bmatrix} = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & d & e \\ \hline 1 & -1 & 1 \\ e & -d & 0 \\ \hline 1 & 1 & 1 \\ 0 & d & e \end{array}\right].$$

For observability, we need to have 3 linearly independent rows or columns $\mathcal{O}$, since the order of the system is 3. Considering the columns of $\mathcal{O}$, if the equation

$$k_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ e \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ d \\ -1 \\ -d \\ 1 \\ d \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ e \\ 1 \\ 0 \\ 1 \\ e \end{bmatrix} = 0$$

has a non-zero solution for k_1, k_2, and k_3; then we don't have 3 linearly independent columns, and the system is not observable. Rewriting the above equation, we get

$$\begin{aligned} k_1 + k_2 + k_3 &= 0, \\ k_2 d + k_3 e &= 0, \\ k_1 - k_2 + k_3 &= 0, \\ k_1 e - k_2 d &= 0, \\ k_1 + k_2 + k_3 &= 0, \\ k_2 d + k_3 e &= 0. \end{aligned}$$

Subtracting the third equation above from the first equation, we get $k_2 = 0$. After substituting $k_2 = 0$ and removing identical equations, we get

$$\begin{aligned} k_1 + k_3 &= 0, \\ k_3 e &= 0, \\ k_1 e &= 0, \end{aligned}$$

or $k_1 e = 0$, and $k_3 = -k_1$.

If $e = 0$; then $k_1 = \alpha$, $k_2 = 0$, and $k_3 = -\alpha$ for any non-zero α form a non-zero solution to the linearly-independence condition, and the system is not observable. However, if $e \neq 0$; then $k_1 = k_2 = k_3 = 0$ is the only solution, and the system is observable.

Reachability: One method to check the reachability of the system is by checking the rank of the controllability matrix

$$\mathcal{C}(A,B) = \left[\begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array}\right].$$

In our case, the system order $n = 3$, and

$$\mathcal{C}(A,B) = \left[\begin{array}{c|c|c} B & AB & A^2B \end{array}\right] = \left[\begin{array}{cc|cc|cc} 1 & b & -1 & c & 1 & b \\ a & 0 & -a & 0 & a & 0 \\ -1 & c & 1 & b & -1 & c \end{array}\right].$$

For reachability, we need to have 3 linearly independent rows or columns $\mathcal{C}$, since the order of the system is 3. Considering the rows of $\mathcal{C}$, if the equation

$$k_1 \begin{bmatrix} 1 \\ b \\ -1 \\ c \\ 1 \\ b \end{bmatrix}^T + k_2 \begin{bmatrix} a \\ 0 \\ -a \\ 0 \\ a \\ 0 \end{bmatrix}^T + k_3 \begin{bmatrix} -1 \\ c \\ 1 \\ b \\ -1 \\ c \end{bmatrix}^T = 0$$

has a non-zero solution for k_1, k_2, and k_3; then we don't have 3 linearly independent columns, and the system is not reachable. Rewriting the above equation, we get

$$\begin{aligned} k_1 + k_2 a - k_3 &= 0, \\ k_1 b + k_3 c &= 0, \\ -k_1 - k_2 a + k_3 &= 0, \\ k_1 c + k_3 b &= 0, \\ k_1 + k_2 a - k_3 &= 0, \\ k_1 b + k_3 c &= 0. \end{aligned}$$

After removing the identical equation, we have

$$\begin{aligned} k_1 \;+ k_2 a - k_3 \;&= 0, \\ k_1 b \qquad\quad + k_3 c &= 0, \\ k_1 c \qquad\quad + k_3 b &= 0. \end{aligned}$$

Multiplying the first equation by $(b-c)$ and subtracting it from the difference of the second and third equations, we get

$$k_2 a(b-c) = 0.$$

If $a = 0$ or $b = c$; then k_2 could be any non-zero value, and there will be a non-zero solution to the linearly-independence condition. So, we need to have $a \neq 0$ and $b \neq c$ as one of the conditions for reachability.

Multiplying the second equation by b, then multiplying the third equation by $-c$, and adding the two products together, we get

$$k_1(b^2 - c^2) = 0.$$

We also need to have $b^2 \neq c^2$ for a zero solution to the linearly-independence condition.

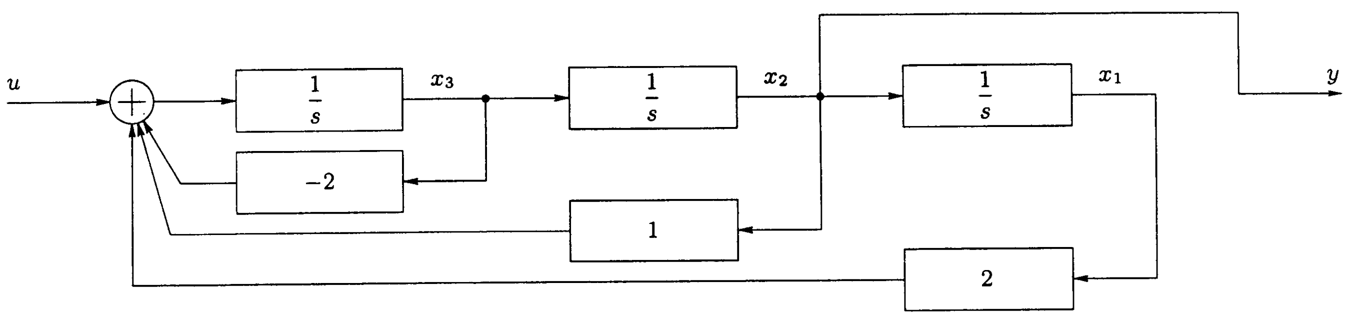
In a similar way, multiplying the second equation by $-c$, then multiplying the third equation by b, and adding the two products together, we get

$$k_3(b^2 - c^2) = 0.$$

Hence, if $a \neq 0$ and $b^2 \neq c^2$; the only solution to the linearly-independence condition is the zero solution, and the system is reachable.

Combining the conditions for observability and reachability, we conclude that the system is minimal, if and only if $a \neq 0$, $b^2 \neq c^2$, and $e \neq 0$.

4. The block diagram of a control system is given below.



Design a proportional-integral-derivative (PID) controller

$$u(t) = K_P y(t) + K_I \int^t y(\tau)\,d\tau + K_D \dot{y}(t),$$

where u and y are the input and the output variables, respectively, such that the closed-loop poles of the system are at $s = -1$ and $s = -2 \pm j1$. Here, K_P, K_I, and K_D are the proportional, the integral, and the derivative constants, respectively. Assume that the internal states x_1, x_2, and x_3 are available to the controller.

Solution: From an inspection of the system block-diagram, we observe that $y=x_2$, $\dot{y}=\dot{x}_2=x_1$, and $\int y(\tau)\,\mathrm{d}\tau=\int x_2(\tau)\,\mathrm{d}\tau=x_3$. So, the PID control is indeed a state-feedback control, where

$$u(t)=K_D x_1(t)+K_P x_2(t)+K_I x_3(t).$$

To design for the control, we write the system equations from the block diagram.

$$\begin{aligned}\dot{x}_1(t)&=x_2(t),\\ \dot{x}_2(t)&=x_3(t),\\ \dot{x}_3(t)&=2x_1(t)+x_2(t)-2x_3(t)+u(t);\end{aligned}$$

or in matrix form

$$\dot{\mathbf{x}}(t)=\begin{bmatrix}0&1&0\\0&0&1\\2&1&-2\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}0\\0\\1\end{bmatrix}u(t),$$

where $\mathbf{x}=\begin{bmatrix}x_1&x_2&x_3\end{bmatrix}^T$. For $u(t)=\begin{bmatrix}K_D&K_P&K_I\end{bmatrix}\mathbf{x}(t)$, we get

$$\dot{\mathbf{x}}(t)=\begin{bmatrix}0&1&0\\0&0&1\\K_D+2&K_P+1&K_I-2\end{bmatrix}\mathbf{x}(t).$$

We also observe that the system is in controller canonical form, and its transfer function can easily be obtained from the last row of the state matrix. So, the characteristic polynomial of the closed-loop system

$$q_{\text{closed-loop}}(s)=s^3-(K_I-2)s^2-(K_P+1)s-(K_D+2).$$

The characteristic polynomial of the desired system is obtained from the desired-pole locations, such that

$$q_{\text{desired}}(s)=\big(s-(-1)\big)\big(s-(-2+j1)\big)\big(s-(-2-j1)\big)=s^3+5s^2+9s+5.$$

Comparing the coefficients of the closed-loop and the desired characteristic polynomials, we get

$$\begin{aligned}K_I-2&=-5,\\ K_P+1&=-9,\\ K_D+2&=-5,\end{aligned}$$

or $K_I=-3$, $K_P=-10$, and $K_D=-7$. Therefore,

$$\begin{aligned}u(t)&=-10y(t)-3\int^t y(\tau)\,\mathrm{d}\tau-7\dot{y}(t)\\ &=-10x_2(t)-3x_3(t)-7x_1(t).\end{aligned}$$