EE 435 | Exam#2 | Dec. 12, 1996

75 minutes

1. A control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{u}(t),$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{u}$, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Determine the initial condition $\mathbf{x}(0)$; if

$$y(0) = 4, \quad \dot{y}(0) = -2, \quad \ddot{y}(0) = -4, \quad \mathbf{u}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \text{ and } \dot{\mathbf{u}}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(30pts)

2. A control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ -1 & 0 \end{bmatrix} \mathbf{u}(t),$$
$$y(t) = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{u}$, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Determine whether or not the system is controllable and/or observable. (20pts)

3. Consider a control system described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \mathbf{x}(t) + B\mathbf{u}(t),$$

where $\mathbf{u}$ and $\mathbf{x}$ are the input and the state variables, respectively, and λ is a real number. Determine the minimal conditions on the elements of B, such that the system is controllable.

(a)

$$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

(10pts)

(b)

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \\ b_{4,1} & b_{4,2} \end{bmatrix}.$$

(10pts)

4. The transfer matrix of a control system is given by

$$H(s) = \begin{bmatrix} \dfrac{s^2}{(s+1)^2(s+2)^2} & \dfrac{s}{(s+1)^2(s+2)} & \dfrac{-s}{(s+1)(s+2)} \\ \dfrac{-1}{(s+2)^2} & \dfrac{1}{(s+1)} & \dfrac{1}{(s+1)^2} \end{bmatrix}.$$

Obtain its left coprime factorization, such that $H = D^{-1}N$ for some matrices D and N, and the matrix N is in Hermite form. (30pts)

#1

$$\dot{x} = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

and
$$y(0) = 4$$
$$\dot{y}(0) = -2 \qquad u(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \dot{u}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\ddot{y}(0) = -4$$

A general expression for $x(0)$ has been derived, but since the derivation was so easy that it will be repeated here for a two dimensional state space representation.

let
$$y = Cx \implies y(0) = Cx(0)$$
$$\dot{y} = C\dot{x}$$
$$= C(Ax + Bu)$$
$$= CAx + CBu \implies \dot{y}(0) = CAx(0) + CBu(0)$$

since A^2 is dependent on I & A from C-H thm, there is no need to go further

so
$$\begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix} x(0) + \begin{bmatrix} 0 \\ CB \end{bmatrix} u(0)$$

$$x(0) = \begin{bmatrix} C \\ \hline CA \end{bmatrix}^{-1}\left(\begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix} - \begin{bmatrix} 0 \\ CB \end{bmatrix} u(0) \right)$$

$$= \begin{bmatrix} 1 & 1 \\ -11 & 1 \end{bmatrix}^{-1}\left(\begin{bmatrix} 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} \right)$$

$$= \frac{1}{12}\begin{bmatrix} 1 & -1 \\ 11 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -8 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 12 \\ 36 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

#2

$$\dot{x} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ -1 & 0 \end{bmatrix} u$$

$$y = (1 \quad 1 \quad -1)\, x$$

Controllability from the rank of $\mathcal{C}(A,B) = [B \mid AB \mid A^2B]$

$$\mathcal{C}(A,B) = \begin{array}{c} \left[\begin{array}{cc|cc|cc} 1 & 2 & 0 & 2 & 0 & 4 \\ -1 & 0 & 0 & 2 & 0 & 4 \\ -1 & 0 & 0 & 2 & 0 & 4 \end{array}\right] \\ \begin{array}{cccccc} a & b & & 2b-2a & & 4b-4a \end{array} \end{array} \leftarrow \text{rank } \mathcal{C}(A,B) = 2$$

NOT CONTROLLABLE

Observability from the rank of $\mathcal{O}(C,A) = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$

$$\mathcal{O}_1^{(C,A)} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$\det \mathcal{O}_1^{(C,A)} = -4 - 3 - 1 - (-2 - 2 - 3) = -1 \neq 0 \rightarrow \text{rank}(\mathcal{O}(C,A)) = 3$$

OBSERVABLE

#3

$$\dot{x} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} x + Bu$$

a) $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ Controllability also from the rank of $[sI - A \quad B]$ for any s

$$[sI - A \quad B] = \left[\begin{array}{cccc|c} s-\lambda & -1 & 0 & 0 & b_1 \\ 0 & s-\lambda & -1 & 0 & b_2 \\ 0 & 0 & s-\lambda & 0 & b_3 \\ 0 & 0 & 0 & s-\lambda & b_4 \end{array}\right]$$

for $s=\lambda$ $= \left[\begin{array}{cccc|c} 0 & -1 & 0 & 0 & b_1 \\ 0 & 0 & -1 & 0 & b_2 \\ 0 & 0 & 0 & 0 & b_3 \\ 0 & 0 & 0 & 0 & b_4 \end{array}\right]$

(since for s not equal to one of the eigenvalues rank of $(sI - A)$ is always n.)

there are only 3 nonzero vectors
i.e. rank can never be 4
so for all choice of B, system is
UNCONTROLLABLE

b) $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}$

Similar to the previous part

$$[sI-A \mid B] = \left[\begin{array}{cccc|cc} s-\lambda & -1 & 0 & 0 & b_{11} & b_{12} \\ 0 & s-\lambda & -1 & 0 & b_{21} & b_{22} \\ 0 & 0 & s-\lambda & 0 & b_{31} & b_{32} \\ 0 & 0 & 0 & s-\lambda & b_{41} & b_{42} \end{array}\right]$$

$$\text{for } s=\lambda = \left[\begin{array}{cccc|cc} 0 & -1 & 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & -1 & 0 & b_{21} & b_{22} \\ 0 & 0 & 0 & 0 & b_{31} & b_{32} \\ 0 & 0 & 0 & 0 & b_{41} & b_{42} \end{array}\right]$$

lin. ind. vectors — possible lin. ind. vectors

rank $[sI-A \mid B] = 4$ if the two new vectors form A lin. independent set

i.e. $\begin{bmatrix} -1 & 0 & b_{11} & b_{12} \\ 0 & -1 & b_{21} & b_{22} \\ 0 & 0 & b_{31} & b_{32} \\ 0 & 0 & b_{41} & b_{42} \end{bmatrix}$ has 4 lin. indep columns

(has form $\begin{bmatrix} E_1 & E_2 \\ 0 & E_3 \end{bmatrix}$)

since the two column span all the elements in the first two locations, the remaining columns should span all the elements in the last two location or

$\begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}$ has rank 2 or $\det\begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} \neq 0$

or $\boxed{b_{31}b_{42} - b_{41}b_{32} \neq 0}$

#4

$$H(s) = \begin{bmatrix} \frac{s^2}{(s+1)^2(s+2)^2} & \frac{s}{(s+1)^2(s+2)} & \frac{-s}{(s+1)(s+2)} \\ \frac{-1}{(s+2)^2} & \frac{1}{s+1} & \frac{1}{(s+1)^2} \end{bmatrix}$$

the gcd is $(s+1)^2(s+2)$, so

$$H(s) = \underbrace{\begin{bmatrix} \frac{1}{(s+1)^2(s+2)^2} & \\ 0 & \frac{1}{(s+1)^2(s+2)^2} \end{bmatrix}}_{D^{-1}(s)} \underbrace{\begin{bmatrix} s^2 & s(s+2) & -s(s+1)(s+2) \\ -(s+1)^2 & (s+1)(s+2)^2 & (s+2)^2 \end{bmatrix}}_{N(s)}$$

Consider $[N(s) \,\vdots\, D(s)]$ and by row operations reduce and put it into the Hermite form.

$$[N(s) \,\vdots\, D(s)]$$

$$= \begin{bmatrix} s^2 & s(s+2) & -s(s+1)(s+2) & (s+1)^2(s+2)^2 & 0 \\ -(s+1)^2 & (s+1)(s+2)^2 & (s+2)^2 & 0 & (s+1)^2(s+2)^2 \end{bmatrix} \begin{matrix} (1) \\ (2) \end{matrix}$$

$$\Big\downarrow \; (1)+(2) \to (4)$$

$$= \begin{bmatrix} s^2 & s(s+2) & -s(s+1)(s+2) & (s+1)^2(s+2)^2 & 0 \\ -2s-1 & (s+2)(s^2+4s+2) & (s+2)(-s^2+2) & (s+1)^2(s+2)^2 & (s+1)^2(s+2)^2 \end{bmatrix} \begin{matrix} (3) \\ (4) \end{matrix}$$

$$\Big\downarrow \; \tfrac{1}{2}s\,(4)+(3) \to (5)$$

$$= \begin{bmatrix} -\frac{s}{2} & \frac{s(s+2)^3}{2} & -\frac{s^2(s+2)^2}{2} & \frac{(s+1)^2(s+2)^3}{2} & \frac{s(s+1)^2(s+2)^2}{2} \\ -2s-1 & (s+2)(s^2+4s+2) & (s+2)(-s^2+2) & (s+1)^2(s+2)^2 & (s+1)^2(s+2)^2 \end{bmatrix} \begin{matrix} (5) \\ (6) \end{matrix}$$

$$\Big\downarrow \; -4(5)+(6) \to (8)$$

$$= \begin{bmatrix} -\frac{s}{2} & \frac{s(s+2)^3}{2} & -\frac{s^2(s+2)^2}{2} & \frac{(s+1)^2(s+2)^3}{2} & \frac{s(s+1)^2(s+2)^2}{2} \\ -1 & -2s^4-11s^3-18s^2-6s+4 & 2s^4+7s^3+6s^2+2s+4 & -(2s+3)(s^2+3s+2)^2 & (-2s+1)(s^2+3s+2) \end{bmatrix} \begin{matrix} (7) \\ (8) \end{matrix}$$

$\downarrow$ $-(8) \to (9)$, $(7) \to (10)$

$$= \begin{bmatrix} 1 & 2s^4+11s^3+18s^2+6s-4 & -2s^4-7s^3-6s^2-2s-4 & (2s+3)(s^2+3s+2)^2 & (2s-1)(s^2+3s+2) \\ -\frac{s}{2} & \frac{s(s+2)^3}{2} & -\frac{s^2(s+2)^2}{2} & \frac{(s+1)^2(s+2)^2}{2} & \frac{s(s+1)^2(s+2)^2}{2} \end{bmatrix} \begin{matrix} (9) \\ (10) \end{matrix}$$

$\downarrow$ $\frac{s}{2}(9) + (10) \to (12)$

$$= \begin{bmatrix} 1 & 2s^4+11s^3+18s^2+6s-4 & -2s^4-7s^3-6s^2-2s-4 & (2s+3)(s^2+3s+2)^2 & (2s-1)(s^2+3s+2) \\ 0 & s(s^4+6s^3+12s^2+9s+2) & -s(s^4+4s^3+5s^2+3s+2) & (s+1)^4(s+2)^2 & s^2(s^2+3s+2)^2 \end{bmatrix} \begin{matrix} (11) \\ (12) \end{matrix}$$

← Hermite Form

$$= \begin{bmatrix} 1 & s(s+2)(s^3+4s^2+4s+1) & -s(s+2)(s^3+2s^2+s+1) & (s+1)(s+2)(2s+3) & (s+1)(s+2)(2s+1) \\ 0 & s(s+1)(s+2)(s^2+3s+1) & -s(s+2)(s^3+2s^2+s+1) & (s+1)^2(s+2)^2 & (s+1)^2(s+2)^2 \end{bmatrix} \begin{matrix} (13) \\ (14) \end{matrix}$$

$\downarrow$ $(14)/(s+2) \to (16)$

$$= \left[\begin{array}{ccc|cc} 1 & s(s+2)(s^3+4s^2+4s+1) & -s(s+2)(s^3+2s^2+s+1) & (s+1)(s+2)(2s+3) & (s+1)(s+2)(2s+1) \\ 0 & s(s+1)(s^2+3s+1) & -s(s^3+2s^2+s+1) & (s+1)^2(s+2) & (s+1)^2(s+2) \end{array}\right] \begin{matrix} (15) \\ (16) \end{matrix}$$

$$\hookrightarrow N(s) = \begin{bmatrix} 1 & s(s+2)(s^3+4s^2+4s+1) & -s(s+2)(s^3+2s^2+s+1) \\ 0 & s(s+1)(s^2+3s+1) & -s(s^3+2s^2+s+1) \end{bmatrix}$$

$$\hookrightarrow D(s) = \begin{bmatrix} (s+1)(s+2)(2s+3) & (s+1)(s+2)(2s+1) \\ (s+1)^2(s+2) & (s+1)^2(s+2) \end{bmatrix}$$

$$\hookrightarrow \det(D(s)) = (s+1)^4(s+2)^3$$

order reduced by one.