EE 431 | Exam#1 | Oct. 19, 2006

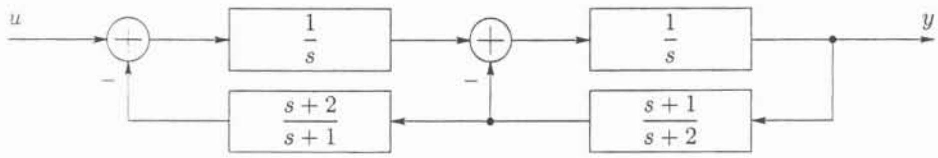
75 minutes

1. Consider the matrix

$$A = \begin{bmatrix} \pi/3 & 0 & a \\ 0 & \pi/3 & b \\ 0 & 0 & \pi/3 \end{bmatrix}.$$

Determine $\cos(A)$. Simplify the expressions as much as possible. (20pts)

2. The block diagram of a control system is given below.



Obtain a state-space representation of the system without any block-diagram reduction. (20pts)

3. A control system is described by

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{2s}{s^2+4s+3} \\ 0 & \dfrac{3}{s+2} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix},$$

where $\boldsymbol{U} = \mathcal{L}\left[\boldsymbol{u}\right] = \left[\begin{array}{cc} U_1 & U_2 \end{array}\right]^T$ and $\boldsymbol{Y} = \mathcal{L}\left[\boldsymbol{y}\right] = \left[\begin{array}{cc} Y_1 & Y_2 \end{array}\right]^T$ are the input and the output variables, respectively. Obtain a state-space representation of the system with no more than three state variables. (20pts)

4. A continuous-time linear control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \end{bmatrix} u(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively.

(a) Determine $y(t)$ for $t \geq 0$, when $\boldsymbol{x}(0) = \left[\begin{array}{cc} -5 & 2 \end{array}\right]^T$, and $u(t) = 0$ for $t \geq 0$. (10pts)

(b) Determine the transfer function of the system. (10pts)

5. A control system is described in state-space representation, such that

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t),$$

where $\boldsymbol{u}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ are the input, the state, and the output variables, respectively. For the following A, B, C, and D matrices, determine whether the system is asymptotically stable, marginally stable, or unstable; and whether it is bounded-input-bounded-output stable or not. Justify your answer.

(a)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and } D = 0.$$

(10pts)

(b)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \text{ and } D = 0.$$

(10pts)

EE 431 **Exam#1** **Solutions** Oct. 19, 2006

1. Consider the matrix

$$A = \begin{bmatrix} \pi/3 & 0 & a \\ 0 & \pi/3 & b \\ 0 & 0 & \pi/3 \end{bmatrix}.$$

Determine $\cos(A)$. Simplify the expressions as much as possible.

Solution: Any function, that has a non-trivial taylor series expansion, of an nth order matrix can be determined either by its taylor series expansion, where

$$f(A) = \sum_{i=0}^{\infty} \frac{1}{i!} \left[\frac{\mathrm{d}^i f(x)}{\mathrm{d}x^i} \right]_{x=0} A^i;$$

by the use of the Cayley-Hamilton's theorem, where

$$f(A) = \sum_{i=0}^{n-1} \alpha_i A^i$$

for some scalars α_i, $i = 0, \ldots, n-1$; or by diagonalization, where

$$f(A) = T f(T^{-1}AT) T^{-1}$$

for a transformation T, such that $T^{-1}AT$ is in jordan form and the evaluation of $f(T^{-1}AT)$ is directly performed.

In the use of the Cayley-Hamilton's theorem, the scalars are determined by the application of the eigenvectors to the above equation that results in the set of equations

$$f(\lambda_1) = \alpha_0 + \alpha_1\lambda_1 + \cdots + \alpha_{n-1}\lambda_1^{n-1}$$
$$\vdots$$
$$f(\lambda_n) = \alpha_0 + \alpha_1\lambda_n + \cdots + \alpha_{n-1}\lambda_n^{n-1},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues, and they are determined from

$$\det\big(\lambda I - A\big) = 0.$$

To be able to solve for all the unknown variables $\alpha_0, \ldots, \alpha_n$; we need to have n linearly independent equations. However, when an eigenvalue is repeated, we get the same equation more than once. In such a case, we use the partial derivatives of the equations for the repeated eigenvalue with respect to the eigenvalue, or

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda_i^k}\Big(f(\lambda_i) = \alpha_0 + \alpha_1\lambda_i + \cdots + \alpha_{n-1}\lambda_i^{n-1} \Big)$$

for $k = 1, \ldots, r$, where r is the number of repetitions of the eigenvalue λ_i. In our case, $n = 3$; so

$$f(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2;$$

and the eigenvalues can be observed to be $\lambda_{1,2,3} = \pi/3$, since the A matrix is upper diagonal. For $f = \cos$, the set of equations becomes

$$\begin{aligned}\cos(\lambda) &= \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2,\\ \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big(\cos(\lambda)\Big) &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\Big)\\ \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\Big(\cos(\lambda)\Big) &= \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\Big(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\Big);\end{aligned}$$

$$\begin{aligned}\cos(\lambda) &= \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2,\\ -\sin(\lambda) &= \alpha_1 + 2\alpha_2\lambda\\ -\cos(\lambda) &= 2\alpha_2;\end{aligned}$$

or for $\lambda = \pi/3$

$$\begin{aligned}1/2 &= \alpha_0 + (\pi/3)\alpha_1 + (\pi^2/9)\alpha_2,\\ -\sqrt{3}/2 &= \alpha_1 + (2\pi/3)\alpha_2\\ -1/2 &= 2\alpha_2.\end{aligned}$$

Solving the above equations simultaneously, we get

$$\alpha_0 = \frac{1}{2} + \frac{\sqrt{3}\pi}{6} - \frac{\pi^2}{36}, \quad \alpha_1 = -\frac{\sqrt{3}}{2} + \frac{\pi}{6}, \text{ and } \alpha_2 = -\frac{1}{4}.$$

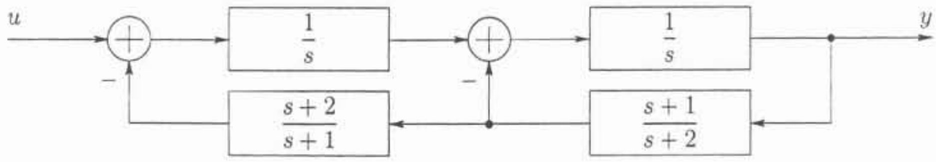
As a result,

$$\begin{aligned}\cos(A) = {}& \left(\frac{1}{2} + \frac{\sqrt{3}\pi}{6} - \frac{\pi^2}{36}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \left(-\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right)\begin{bmatrix} \pi/3 & 0 & a \\ 0 & \pi/3 & b \\ 0 & 0 & \pi/3 \end{bmatrix}\\ &+ \left(-\frac{1}{4}\right)\begin{bmatrix} \pi^2/9 & 0 & 2\pi a/3 \\ 0 & \pi^2/9 & 2\pi b/3 \\ 0 & 0 & \pi^2/9 \end{bmatrix}.\end{aligned}$$

After simplifying the above expression, we get

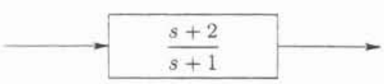
$$\cos(A) = \begin{bmatrix} 1/2 & 0 & -\sqrt{3}a/2 \\ 0 & 1/2 & -\sqrt{3}b/2 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

2. The block diagram of a control system is given below.

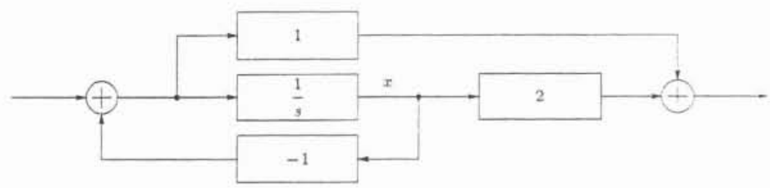


Obtain a state-space representation of the system without any block-diagram reduction.

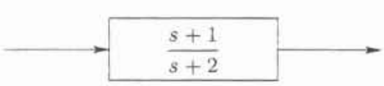
Solution: In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.



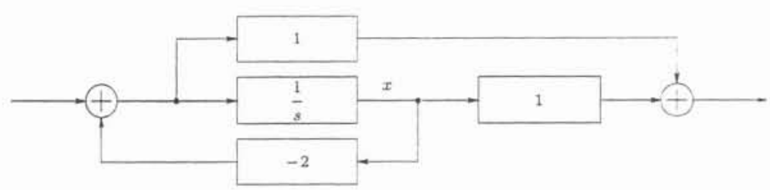
(a) The first feedback gain block.



(b) Controller realization form.

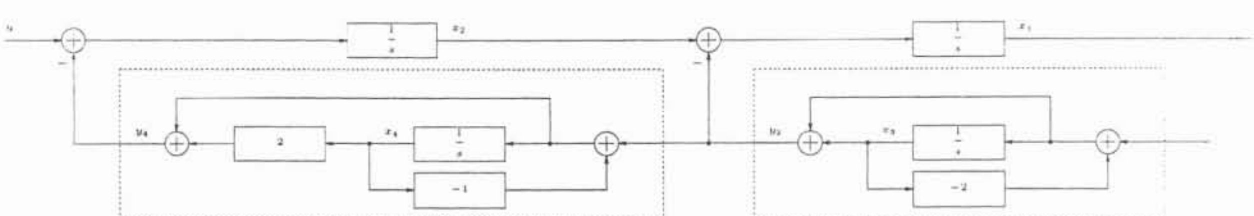


(a) The second feedback gain block.



(b) Controller realization form.

The connected and "expanded" block diagram is shown below.



After assigning the state variables as shown in the figure, we obtain

$$\begin{aligned}
\dot{x}_1 &= x_2 - y_3, \\
\dot{x}_2 &= u - y_4, \\
\dot{x}_3 &= -2x_3 + x_1, \\
\dot{x}_4 &= -x_4 + y_3,
\end{aligned}$$

and

$$y = x_1;$$

where

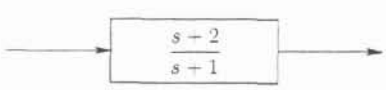
$$\begin{aligned} y_3 &= x_3 + (-2x_3 + x_1) = x_1 - x_3, \\ y_4 &= 2x_4 + (-x_4 + y_3) = x_1 - x_3 + x_4. \end{aligned}$$

After substituting the y_3 and y_4 expressions into the differential equations and writing them in matrix form, we obtain the state-space representation

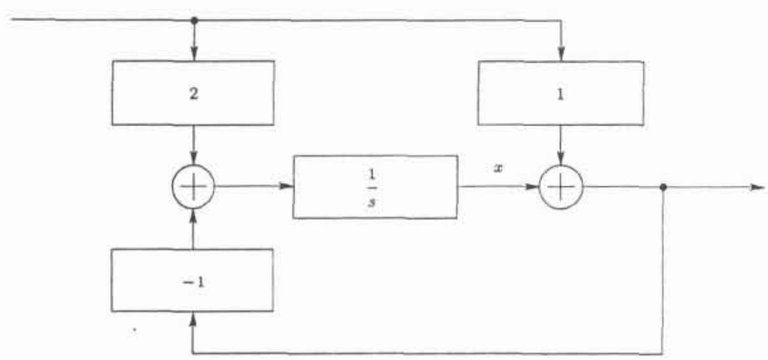
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & -1 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}.$$

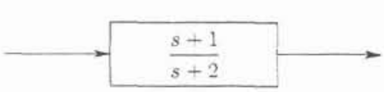
If we use the observer realization form for each of the blocks, then we obtain a different state-space representation.



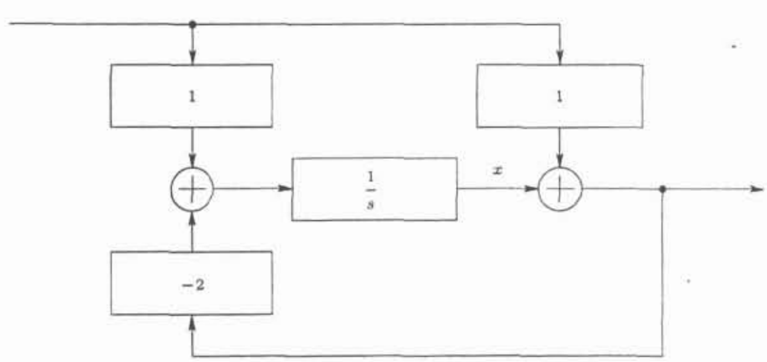
(a) The first feedback gain block.



(b) Observer realization form.

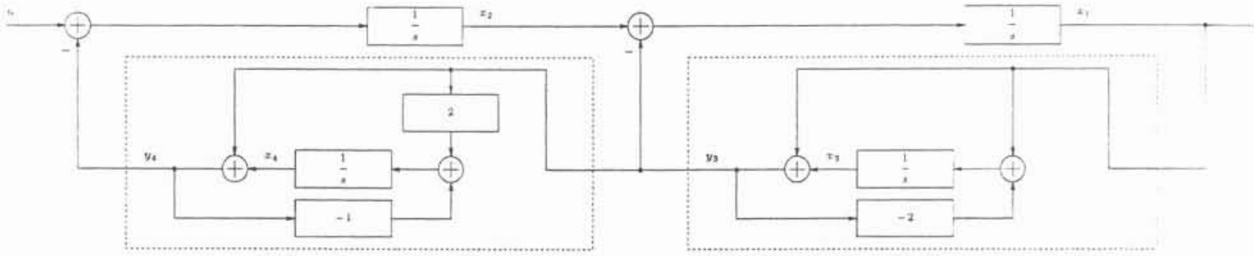


(a) The second feedback gain block.



(b) Observer realization form.

The connected and "expanded" block diagram for this case is shown below.



Similarly, we obtain

$$\begin{aligned}
\dot{x}_1 &= x_2 - y_3, \\
\dot{x}_2 &= u - y_4, \\
\dot{x}_3 &= -2y_3 + x_1, \\
\dot{x}_4 &= -y_4 + 2y_3,
\end{aligned}$$

and

$$y = x_1;$$

where

$$\begin{aligned}
y_3 &= x_3 + x_1, \\
y_4 &= x_4 + y_3 = x_1 + x_3 + x_4.
\end{aligned}$$

After substituting the y_3 and y_4 expressions into the differential equations and writing them in matrix form, we obtain another state-space representation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 & 0 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -2 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}.$$

3. A control system is described by

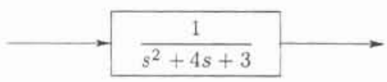
$$\begin{bmatrix} Y_1(s) \\ \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{2s}{s^2+4s+3} \\ \\ 0 & \dfrac{3}{s+2} \end{bmatrix} \begin{bmatrix} U_1(s) \\ \\ U_2(s) \end{bmatrix},$$

where $\boldsymbol{U} = \mathcal{L}\left[\boldsymbol{u}\right] = \left[\, U_1 \;\; U_2 \,\right]^T$ and $\boldsymbol{Y} = \mathcal{L}\left[\boldsymbol{y}\right] = \left[\, Y_1 \;\; Y_2 \,\right]^T$ are the input and the output variables, respectively. Obtain a state-space representation of the system with no more than three state variables.

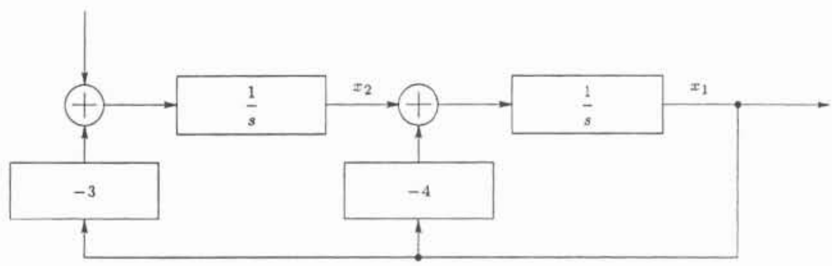
Solution: From the transfer matrix, we observe that even though we have two inputs and two outputs. the second output is decoupled from the dynamics of the first output; since the poles are distinct. As a result, we may be able to realize the two output dynamics independently and factor out the common denominator for the first output.

$$Y_1(s) = \Big[(s+1)(s+3)\Big]^{-1}\begin{bmatrix} s+3 & 2s \end{bmatrix}\begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} = \Big[s^2+4s+3\Big]^{-1}\begin{bmatrix} s+3 & 2s \end{bmatrix}\begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}.$$

When the denominator polynomial is realized in the observer realization form, the two elements in the numerator matrix can be generated independently as the feedforward terms in the observer realization form.

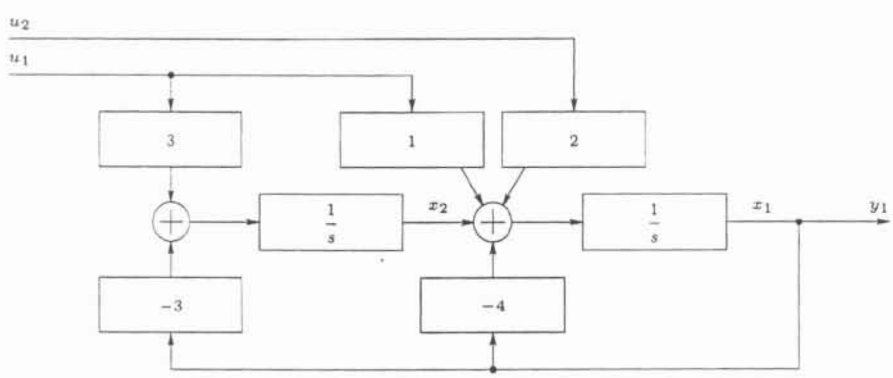


(a) The feedback portion.

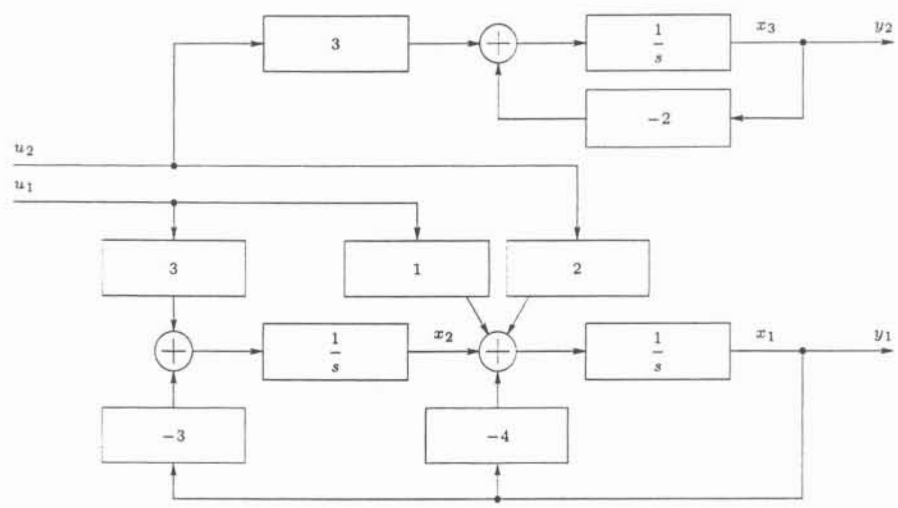


(b) Observer realization form.

When the two elements in the numerator matrix are generated as the feedforward terms, we get the following block diagram.



The realization of the second output is straightforward; and when it is combined with the first output, we get the final form of the realization.



Next, with the assignment of the state variables as shown in the figure we obtain

$$\dot{x}_1 = -4x_1 + x_2 + u_1 + 2u_2,$$
$$\dot{x}_2 = -3x_1 + 3u_1,$$
$$\dot{x}_3 = -2x_3 + 3u_2,$$

and

$$y_1 = x_1,$$
$$y_2 = x_3.$$

After expressing the above equations in matrix form, we get the state-space representation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

4. A continuous-time linear control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \end{bmatrix} u(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively.

(a) Determine $y(t)$ for $t \geq 0$, when $\boldsymbol{x}(0) = \begin{bmatrix} -5 & 2 \end{bmatrix}^T$, and $u(t) = 0$ for $t \geq 0$.

Solution: The general solution to the state-space representation of a system described by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t)$$

is obtained from

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)} B\boldsymbol{u}(\tau)\,\mathrm{d}\tau,$$

where

$$e^{At} = \mathcal{L}_s^{-1}\left[(sI - A)^{-1}\right](t).$$

Here, I is the appropriately dimensioned identity matrix. In our case,

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 \end{bmatrix},$$

and $u(t) = 0$ for $t \geq 0$. As a result, the integral term in the solution of x is identically zero.

To determine e^{At}, we may use a few different methods. However in this case, we will use the method based on the Cayley-Hamilton theorem. In this method, we observe that e^{At} may be described by a linear combination of A^k for $k = 0, \ldots, (n-1)$, so that

$$e^{At} = \alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1}A^{n-1},$$

where I is the appropriately dimensioned identity matrix, n is the dimension of the system, and $\alpha_0, \ldots, \alpha_{n-1}$ are scalars. The scalars are determined by the application of the eigenvectors to the above equation that results in the set of equations

$$e^{\lambda_1 t} = \alpha_0 + \alpha_1\lambda_1 + \cdots + \alpha_{n-1}\lambda_1^{n-1},$$
$$\vdots$$
$$e^{\lambda_n t} = \alpha_0 + \alpha_1\lambda_n + \cdots + \alpha_{n-1}\lambda_n^{n-1},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues. To be able to solve for all the unknown variables $\alpha_0, \ldots, \alpha_n$; we need to have n linearly independent equations. However, when an eigenvalue is repeated, we get the same equation more than once. In such a case, we use the partial derivatives of the equations for the repeated eigenvalue with respect to the eigenvalue, or

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda_i^k}\left(e^{\lambda_i t} = \alpha_0 + \alpha_1\lambda_i + \cdots + \alpha_{n-1}\lambda_i^{n-1}\right)$$

for $k = 1, \ldots, r$, where r is the number of repetitions of the eigenvalue λ_i. In our case, $n = 2$, so

$$e^{At} = \alpha_0 I + \alpha_1 A;$$

and the eigenvalues are calculated from

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda & -2 \\ 2 & \lambda \end{bmatrix} = \lambda^2 + 4 = 0,$$

$\frac{s}{s^2+4}$ $\frac{-2}{s^2+4}$

$\frac{8}{s^2+4}$ $\frac{s}{s^2+4}$

or $\lambda_{1,2} = \pm j2$. The set of equations becomes

$$\left[\, e^{\lambda t} \,\right]_{\lambda=+j2} = [\, \alpha_0 + \alpha_1 \lambda \,]_{\lambda=+j2},$$

$$\left[\, e^{\lambda t} \,\right]_{\lambda=-j2} = [\, \alpha_0 + \alpha_1 \lambda \,]_{\lambda=-j2}.$$

In our case,

$$e^{(j2)t} = \alpha_0 + \alpha_1(j2),$$
$$e^{(-j2)t} = \alpha_0 + \alpha_1(-j2).$$

Solving the above set of equations simultaneously gives

$$\alpha_0 = \left(e^{j2t} + e^{-j2t}\right)/2 = \cos(2t),$$
$$\alpha_1 = \left(e^{j2t} - e^{-j2t}\right)/(4j) = (1/2)\sin(2t).$$

As a result,

$$e^{At} = \alpha_0 I + \alpha_1 A = \cos(2t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (1/2)\sin(2t)\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix}.$$

Since $u(t) = 0$ for $t \geq 0$, we have

$$y(t) = Ce^{At}\boldsymbol{x}(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix}\begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\cos(2t) - \sin(2t)\right) & \left(\cos(2t) + \sin(2t)\right) \end{bmatrix}\begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

or

$$y(t) = -3\cos(2t) + 7\sin(2t) \text{ for } t \geq 0.$$

(b) Determine the transfer function of the system.

Solution: The transfer function of a control system described in the state-state representation

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$

$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t),$$

is

$$F(s) = C(sI - A)^{-1}B + D.$$

Therefore,

$$F(s) = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} s & -2 \\ 2 & s \end{bmatrix}^{-1}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1 = \left(\frac{1}{s^2+4}\right)\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} s & 2 \\ -2 & s \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1$$

$$= \left(\frac{1}{s^2+4}\right)\begin{bmatrix} s-2 & s+2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1 = \left(\frac{1}{s^2+4}\right)\begin{bmatrix} 4 \end{bmatrix} + 1 = \frac{4}{s^2+4} + 1.$$

In other words, the transfer function is $F(s) = (s^2+8)/(s^2+4)$.

5. A control system is described in state-space representation, such that

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t),$$

where $\boldsymbol{u}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ are the input, the state, and the output variables, respectively. For the following A, B, C, and D matrices, determine whether the system is asymptotically stable, marginally stable, or unstable; and whether it is bounded-input-bounded-output stable or not. Justify your answer.

(a)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and } D = 0.$$

Solution: In order to determine the stability of the system, we first need to determine its eigenvalues or poles. Since in this case, the state matrix A is diagonal, we observe the eigenvalues directly from the diagonal elements as $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = -1$. The eigenvalue λ_3 has a negative real part, and it would generate an asymptotically stable response. The eigenvalues λ_1 and λ_2 are both zero, and each would generate a constant response individually. However, if they are cascaded, a constant response generated by the first one would result in a ramp response by the second one. In this case, since the state matrix A is diagonal, the two zero-valued eigenvalues don't affect each other, or they are not cascaded. Therefore, each of the eigenvalues λ_1 and λ_2 would generate a constant response resulting in a marginally stable response. Since there are no more eigenvalues, we conclude that the system is marginally stable.

In order to determine the bounded-input-bounded-output stability of the system, we may determine the transfer matrix and observe the poles of the system after all the reductions. The transfer matrix of the system is given by

$$\mathcal{L}\left[\boldsymbol{y}\right](s) = \left(C(sI - A)^{-1}B + D\right)\mathcal{L}\left[\boldsymbol{u}\right](s)$$

where $\mathcal{L}\left[(\cdot)\right](s)$ is the Laplace transform, and I is the appropriately dimensioned identity matrix. In our case,

$$\begin{aligned} C(sI - A)^{-1}B + D &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/s & 0 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1/(s+1) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/s \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

We realize that the transfer matrix elements are all zero, therefore the system is bounded-input-bounded-output stable.

In summary, the system is marginally stable, and it is bounded-input-bounded-output stable.

(b)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \text{ and } D = 0.$$

Solution: Since the state matrix A is upper diagonal, we observe the eigenvalues directly from the diagonal elements as $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = -1$. The eigenvalue λ_3 has a negative real part, and it would generate an asymptotically stable response. The eigenvalues λ_1 and λ_2 are both zero, and each would generate a constant response individually. However, if they are cascaded, a constant response generated by the first one would result in a ramp response by the second one. In this case, since the state matrix A is in jordan form, the two zero-valued eigenvalues are cascaded, and they affect each other. Therefore, the state corresponding to λ_2 would generate a constant response, and the state corresponding to λ_1 would then generate a ramp response. As a result, we conclude that the system is unstable.

In order to determine the bounded-input-bounded-output stability of the system, we may determine the transfer matrix and observe the poles of the system after all the reductions. The transfer matrix of the system is given by

$$\mathcal{L}\left[\boldsymbol{y}\right](s) = \left(C(sI-A)^{-1}B + D\right)\mathcal{L}\left[\boldsymbol{u}\right](s)$$

where $\mathcal{L}\left[(\cdot)\right](s)$ is the Laplace transform, and I is the appropriately dimensioned identity matrix. One method to determine the inverse of $(sI - A)$ is to use row operations on the augmented matrix $\left[\ (sI-A) \quad I\ \right]$ to generate $\left[\ I \quad (sI-A)^{-1}\ \right]$.

$$\left[\begin{array}{ccc|ccc} s & -1 & 0 & 1 & 0 & 0 \\ 0 & s & 0 & 0 & 1 & 0 \\ 0 & 0 & s+1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} s & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/s & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/(s+1) \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} s & 0 & 0 & 1 & 1/s & 0 \\ 0 & 1 & 0 & 0 & 1/s & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/(s+1) \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/s & 1/s^2 & 0 \\ 0 & 1 & 0 & 0 & 1/s & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/(s+1) \end{array}\right].$$

Therefore,

$$C(sI-A)^{-1}B + D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/s & 1/s^2 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1/(s+1) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/s \\ 0 \\ 1/(s+1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1/(s+1) \end{bmatrix}.$$

We realize that only one pole of the system is visible in the transfer function; and it has a negative real part. Therefore, the system is unstable, and it is bounded-input-bounded-output stable.