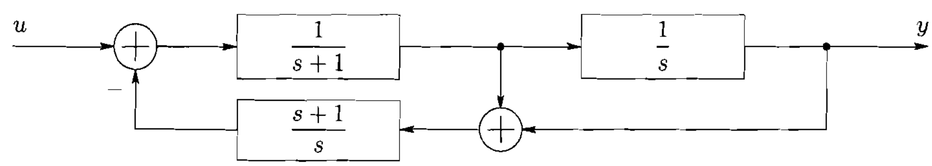
EE 431 Exam#1 Oct. 23, 2007

75 minutes

1. The block diagram of a control system is given below.



Obtain a state-space representation of the system without any block-diagram reduction. (20pts)

2. The state-space equations of a control system are given by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x}(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively.

(a) Determine $y(t)$ for $t \geq 0$; when $\boldsymbol{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, and $u(t) = e^{-t}$ for $t \geq 0$. Show all your work. (15pts)

(b) Determine the transfer function of the system. (10pts)

3. A time-varying linear control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -6t - 5\sin(t) & -3t - 3\sin(t) \\ 10t + 10\sin(t) & 5t + 6\sin(t) \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t),$$

where u and $\boldsymbol{x}$ are the input and the state variables, respectively. Determine $\boldsymbol{x}(t)$ for $t \geq 0$, when $\boldsymbol{x}(0) = \begin{bmatrix} 2 & -3 \end{bmatrix}^T$, and $u(t) = 0$ for $t \geq 0$. (25pts)

4. A control system is described in state-space representation, such that

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t),$$

where $\boldsymbol{u}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ are the input, the state, and the output variables, respectively. For the following A, B, C, and D matrices, determine whether the system is asymptotically stable, marginally stable, or unstable; and whether it is bounded-input-bounded-output stable or not. Justify your answer.

(a)

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and } D = 0.$$

(10pts)

(b)

$$A = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and } D = 0.$$

(10pts)

(c)

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } D = 0.$$

(10pts)

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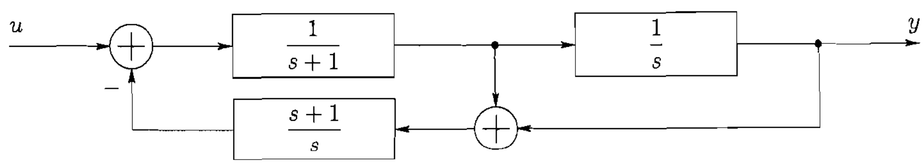
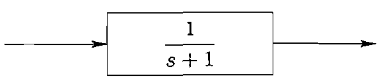
1. The block diagram of a control system is given below.

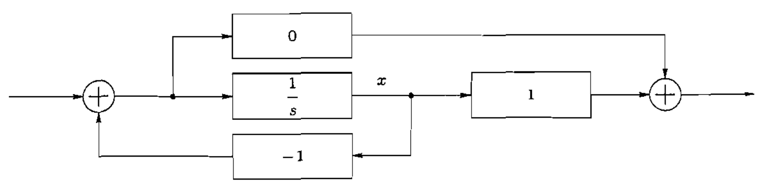


Obtain a state-space representation of the system without any block-diagram reduction.

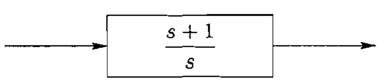
Solution: In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.



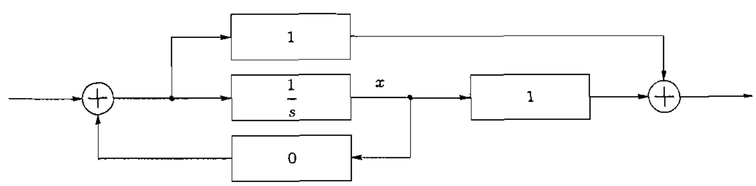
(a) The feedforward gain block.



(b) Controller realization form.

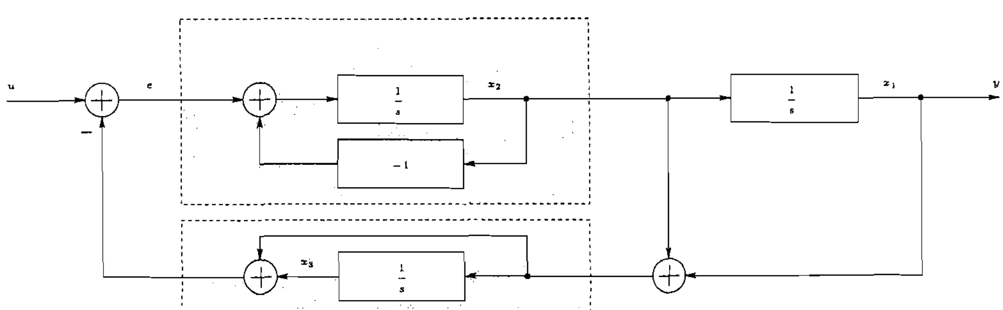


(c) The feedback gain block.



(d) Controller realization form.

The connected and "expanded" block diagram is shown below.



After assigning the state variables as shown in the figure, we obtain

$$\dot{x}_1 = \dot{x}_2,$$
$$\dot{x}_2 = -x_2 + e,$$
$$\dot{x}_3 = x_1 + x_2,$$

and

$$y = x_1.$$

We also have

$$e = u - \left(x_3 + \dot{x}_3\right) = -x_1 - x_2 - x_3 + u.$$

After substituting the expression for e into the original set, we obtain the state-space representation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

In this case, the observer realization form gives the same state-space representation.

2. The state-space equations of a control system are given by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x}(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively.

(a) Determine $y(t)$ for $t \geq 0$; when $\boldsymbol{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, and $u(t) = e^{-t}$ for $t \geq 0$. Show all your work.

Solution: The general solution to the state-space representation of a system described by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t)$$

is obtained from

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)} B\boldsymbol{u}(\tau)\, \mathrm{d}\tau,$$

where

$$e^{At} = \mathcal{L}_s^{-1}\left[(sI - A)^{-1}\right](t).$$

Here, I is the appropriately dimensioned identity matrix. In our case,

$$A = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \end{bmatrix},$$

$\boldsymbol{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, and $u(t) = 1$ for $t \geq 0$. As a result, the initial-condition term in the solution of $\boldsymbol{x}$ is identically zero. So,

$$y(t) = C\int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau + Du(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\int_0^t \mathfrak{L}_s^{-1}\left[(sI-A)^{-1}\right](t-\tau)\begin{bmatrix} 2 \\ 1 \end{bmatrix}\left(e^{-\tau}\right)\,\mathrm{d}\tau$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix}\int_0^t \mathfrak{L}_s^{-1}\left[\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}\right)^{-1}\right](t-\tau)\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-\tau}\,\mathrm{d}\tau$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix}\int_0^t \mathfrak{L}_s^{-1}\left[\left(\begin{bmatrix} s+2 & -1 \\ -2 & s+3 \end{bmatrix}\right)^{-1}\right](t-\tau)\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-\tau}\,\mathrm{d}\tau$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix}\int_0^t \mathfrak{L}_s^{-1}\left[\tfrac{1}{s^2+5s+4}\begin{bmatrix} s+3 & 1 \\ 2 & s+2 \end{bmatrix}\right](t-\tau)\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-\tau}\,\mathrm{d}\tau$$

$$= \int_0^t \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \mathfrak{L}_s^{-1}\left[\frac{s+3}{(s+1)(s+4)}\right](t-\tau) & \mathfrak{L}_s^{-1}\left[\frac{1}{(s+1)(s+4)}\right](t-\tau) \\ \mathfrak{L}_s^{-1}\left[\frac{2}{(s+1)(s+4)}\right](t-\tau) & \mathfrak{L}_s^{-1}\left[\frac{s+2}{(s+1)(s+4)}\right](t-\tau) \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-\tau}\,\mathrm{d}\tau$$

$$= \int_0^t \begin{bmatrix} \mathfrak{L}_s^{-1}\left[\frac{2}{(s+1)(s+4)}\right](t-\tau) & \mathfrak{L}_s^{-1}\left[\frac{s+2}{(s+1)(s+4)}\right](t-\tau) \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-\tau}\,\mathrm{d}\tau$$

$$= \int_0^t \left(2\mathfrak{L}_s^{-1}\left[\tfrac{2}{(s+1)(s+4)}\right](t-\tau) + \mathfrak{L}_s^{-1}\left[\tfrac{s+2}{(s+1)(s+4)}\right](t-\tau)\right)e^{-\tau}\,\mathrm{d}\tau$$

$$= \int_0^t \left(\mathfrak{L}_s^{-1}\left[\tfrac{s+6}{(s+1)(s+4)}\right](t-\tau)\right)e^{-\tau}\,\mathrm{d}\tau = \int_0^t \left(\mathfrak{L}_s^{-1}\left[\tfrac{5/3}{s+1} - \tfrac{2/3}{s+4}\right](t-\tau)\right)e^{-\tau}\,\mathrm{d}\tau$$

$$= \int_0^t \left((5/3)e^{-(t-\tau)} - (2/3)e^{-4(t-\tau)}\right)e^{-\tau}\,\mathrm{d}\tau = \int_0^t \left((5/3)e^{-t} - (2/3)e^{-4t+3\tau}\right)\,\mathrm{d}\tau$$

$$= \left((5/3)e^{-t}\tau - (2/3)(1/3)e^{-4t+3\tau}\right)_{\tau=0}^{\tau=t}.$$

Or,

$$y(t) = \left((5/3)t - (2/9)\right)e^{-t} + (2/9)e^{-4t} \text{ for } t \geq 0.$$

(b) Determine the transfer function of the system.

Solution: The transfer function of a control system described in the state-state representation

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t),$$

is

$$F(s) = C(sI - A)^{-1}B + D;$$

where in our case

$$A = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \end{bmatrix},$$

Here, I is the appropriately dimensioned identity matrix.

Therefore,

$$\begin{aligned} F(s) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+2 & -1 \\ -2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \\ &= \frac{1}{s^2+5s+4} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+3 & 1 \\ 2 & s+2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2+5s+4} \begin{bmatrix} 2 & s+2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{s^2+5s+4} \begin{bmatrix} s+6 \end{bmatrix}. \end{aligned}$$

In other words, the transfer function is $F(s) = \left(s+6\right)/\left(s^2+5s+4\right) = \left(s+6\right)/\left((s+1)(s+4)\right)$.

3. A time-varying linear control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -6t - 5\sin(t) & -3t - 3\sin(t) \\ 10t + 10\sin(t) & 5t + 6\sin(t) \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t),$$

where u and $\boldsymbol{x}$ are the input and the state variables, respectively. Determine $\boldsymbol{x}(t)$ for $t \geq 0$, when $\boldsymbol{x}(0) = \begin{bmatrix} 2 & -3 \end{bmatrix}^T$, and $u(t) = 0$ for $t \geq 0$.

Solution: Since the given system is time varying, the solution is given by

$$\boldsymbol{x}(t) = \Phi(t,0)\boldsymbol{x}(0) + \int_0^t \Phi(t,\tau)B(\tau)\boldsymbol{u}(\tau)\,\mathrm{d}\tau,$$

where Φ is the state-transition matrix, and B is the input matrix. When the state matrix A and its integral commute, or when the commutativity condition $A(t_1)A(t_2) = A(t_2)A(t_1)$ for all t_1 and t_2 is satisfied; the state-transition matrix is given by

$$\Phi(t,t_0) = e^{\left(\int_{t_0}^t A(\tau)\,\mathrm{d}\tau\right)}.$$

In our case,

$$\begin{aligned} A(t_1)A(t_2) &= \begin{bmatrix} -6t_1 - 5\sin(t_1) & -3t_1 - 3\sin(t_1) \\ 10t_1 + 10\sin(t_1) & 5t_1 + 6\sin(t_1) \end{bmatrix} \begin{bmatrix} -6t_2 - 5\sin(t_2) & -3t_2 - 3\sin(t_2) \\ 10t_2 + 10\sin(t_2) & 5t_2 + 6\sin(t_2) \end{bmatrix} \\ &= \begin{bmatrix} 6t_1t_2 - 5\sin(t_1)\sin(t_2) & 3t_1t_2 - 3\sin(t_1)\sin(t_2) \\ -10t_1t_2 + 10\sin(t_1)\sin(t_2) & -5t_1t_2 + 6\sin(t_1)\sin(t_2) \end{bmatrix} = A(t_2)A(t_1), \end{aligned}$$

and the state-transition matrix is exponential of the integral of the state matrix. However, direct computation of the exponential is rather involved and a simpler approach is preferable.

Diagonalization of $A(t)$

One possible approach is to diagonalize the state matrix. If there exists a matrix T, such that

$$T^{-1}A(t)T = \Lambda(t)$$

is diagonal, then

$$\Phi(t,t_0) = e^{\left(\int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau\right)} = Te^{\left(\int_{t_0}^{t} T^{-1}A(\tau)T\,\mathrm{d}\tau\right)}T^{-1} = Te^{\left(\int_{t_0}^{t} \Lambda_A(\tau)\,\mathrm{d}\tau\right)}T^{-1}.$$

To diagonalize the state matrix, we first need to find its eigenvalues from the characteristic equation

$$\det\bigl(\lambda I - A(t)\bigr) = 0,$$

or

$$\det\begin{bmatrix} \lambda + 6t + 5\sin(t) & 3t + 3\sin(t) \\ -10t - 10\sin(t) & \lambda - 5t - 6\sin(t) \end{bmatrix} = \lambda^2 + \bigl(t - \sin(t)\bigr)\lambda + t\sin(t) = \bigl(\lambda + t\bigr)\bigl(\lambda - \sin(t)\bigr) = 0.$$

So, the eigenvalues are $\lambda_1 = -t$ and $\lambda_2 = \sin(t)$.

For $\lambda_1 = -t$, we have

$$\bigl(-tI - A\bigr)\boldsymbol{v}_1 = \begin{bmatrix} 5t + 5\sin(t) & 3t + 3\sin(t) \\ -10t - 10\sin(t) & -6t - 6\sin(t) \end{bmatrix}\begin{bmatrix} v_{1_1} \\ v_{1_2} \end{bmatrix} = 0,$$

or

$$\bigl(5t + 5\sin(t)\bigr)v_{1_1} + \bigl(3t + 3\sin(t)\bigr)v_{1_2} = 0.$$

Letting $v_{1_1} = 3$, we get $\boldsymbol{v}_1 = \begin{bmatrix} 3 & -5 \end{bmatrix}^T$.

Similarly, for $\lambda_2 = \sin(t)$, we have

$$\bigl(\sin(t)I - A\bigr)\boldsymbol{v}_2 = \begin{bmatrix} 6t + 6\sin(t) & 3t + 3\sin(t) \\ -10t - 10\sin(t) & -5t - 5\sin(t) \end{bmatrix}\begin{bmatrix} v_{2_1} \\ v_{2_2} \end{bmatrix} = 0,$$

or

$$\bigl(6t + 6\sin(t)\bigr)v_{2_1} + \bigl(3t + 3\sin(t)\bigr)v_{2_2} = 0.$$

Letting $v_{2_2} = 2$, we get $\boldsymbol{v}_2 = \begin{bmatrix} -1 & 2 \end{bmatrix}^T$.

Therefore,

$$T = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

diagonalizes the state matrix, such that

$$T^{-1}A(t)T = \Lambda_A(t) = \begin{bmatrix} -t & 0 \\ 0 & \sin(t) \end{bmatrix}.$$

First taking the integral of Λ_A, we get

$$\int_{t_0}^{t} \Lambda_A(\tau)\,\mathrm{d}\tau = \begin{bmatrix} \int_{t_0}^{t}(-\tau)\,\mathrm{d}\tau & 0 \\ 0 & \int_{t_0}^{t}(\sin(\tau))\,\mathrm{d}\tau \end{bmatrix}$$

$$= \begin{bmatrix} -\tau^2/2 & 0 \\ 0 & -\cos(\tau) \end{bmatrix}_{\tau=t_0}^{\tau=t} = \begin{bmatrix} -(t^2 - {t_0}^2) & 0 \\ 0 & -\cos(t) + \cos(t_0) \end{bmatrix};$$

second taking the exponential of the integral, we get

$$e^{\left(\int_{t_0}^{t} \Lambda_A(\tau)\,\mathrm{d}\tau\right)} = \begin{bmatrix} e^{-(t^2-t_0{}^2)} & 0 \\ 0 & e^{-\cos(t)+\cos(t_0)} \end{bmatrix};$$

and finally,

$$\begin{aligned}\Phi(t,0) &= Te^{\left(\int_0^t \Lambda_A(\tau)\,\mathrm{d}\tau\right)}T^{-1} \\ &= \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} e^{-t^2/2} & 0 \\ 0 & e^{1-\cos(t)} \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 6e^{-t^2/2} - 5e^{1-\cos(t)} & 3e^{-t^2/2} - 3e^{1-\cos(t)} \\ -10e^{-t^2/2} + 10e^{1-\cos(t)} & -5e^{-t^2/2} + 6e^{1-\cos(t)} \end{bmatrix}.\end{aligned}$$

Since $\boldsymbol{x}(0) = \begin{bmatrix} 2 & -3 \end{bmatrix}^T$, and $u(t) = 0$ for $t \geq 0$; we have $\boldsymbol{x}(t) = \Phi(t,0)\boldsymbol{x}(0)$, or

$$\boldsymbol{x}(t) = \begin{bmatrix} 3e^{-t^2/2} - e^{1-\cos(t)} \\ -5e^{-t^2/2} + 2e^{1-\cos(t)} \end{bmatrix} \text{ for } t \geq 0.$$

Using the Cayley-Hamilton's theorem

Another possible approach is to use the Cayley-Hamilton's theorem to determine $\Phi(t,t_0)$ from the eigenvalues of $A(t)$. In this method, we observe that $\Phi(t,t_0)$ may be described by a linear combination of $A^k(t)$ for $k = 0, \ldots, (n-1)$, so that

$$e^{\left(\int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau\right)} = \alpha_0(t)I + \alpha_1(t)A(t) + \cdots + \alpha_{n-1}(t)A^{n-1}(t),$$

where I is the appropriately dimensioned identity matrix, n is the dimension of the system, and $\alpha_0(t), \ldots, \alpha_{n-1}(t)$ are functions of time. The scalars are determined by the application of the eigenvectors to the above equation that results in the set of equations

$$\begin{aligned} e^{\left(\int_0^t \lambda_1(\tau)\,\mathrm{d}\tau\right)} &= \alpha_0(t)I + \alpha_1(t)\lambda_1(t) + \cdots + \alpha_{n-1}(t)\lambda_1^{n-1}(t) \\ &\vdots \\ e^{\left(\int_0^t \lambda_n(\tau)\,\mathrm{d}\tau\right)} &= \alpha_0(t)I + \alpha_1(t)\lambda_n(t) + \cdots + \alpha_{n-1}(t)\lambda_n^{n-1}(t), \end{aligned}$$

where $\lambda_1(t), \ldots, \lambda_n(t)$ are the eigenvalues. Since we have already determined the eigenvalues of $A(t)$ as $\lambda_1(t) = -t$ and $\lambda_2(t) = \sin(t)$, and $n = 2$; the set of equations becomes

$$\begin{aligned} e^{\left(\int_0^t (-\tau)\,\mathrm{d}\tau\right)} &= \alpha_0(t) + \alpha_1(t)(-t) \\ e^{\left(\int_0^t (\sin(\tau))\,\mathrm{d}\tau\right)} &= \alpha_0(t) + \alpha_1(t)(\sin(t)), \end{aligned}$$

or

$$\begin{aligned} e^{-t^2/2} &= \alpha_0(t) - t\alpha_1(t) \\ e^{1-\cos(t)} &= \alpha_0(t) + \sin(t)\alpha_1(t). \end{aligned}$$

Solving the above set of equations simultaneously gives

$$\alpha_0(t) = \left(te^{1-\cos(t)} + \sin(t)e^{t^2/2}\right) \Big/ \left(t + \sin(t)\right)$$
$$\alpha_1(t) = \left(e^{1-\cos(t)} - e^{t^2/2}\right) \Big/ \left(t + \sin(t)\right)$$

As a result,

$$\Phi(t,0) = \alpha_0(t)I + \alpha_1(t)A(t),$$

and

$$\begin{aligned}
\boldsymbol{x}(t) &= \Phi(t,0)\boldsymbol{x}(0) = \alpha_0(t)\boldsymbol{x}(0) + \alpha_1(t)A(t)\boldsymbol{x}(0) \\
&= \left(\frac{te^{1-\cos(t)} + \sin(t)e^{t^2/2}}{t+\sin(t)}\right)\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \left(\frac{e^{1-\cos(t)} - e^{t^2/2}}{t+\sin(t)}\right)\begin{bmatrix} -3t - \sin(t) \\ 5t + 2\sin(t) \end{bmatrix} \\
&= \left(\frac{1}{t+\sin(t)}\right)\begin{bmatrix} \left(-t-\sin(t)\right)e^{1-\cos(t)} + \left(3t + 3\sin(t)\right)e^{t^2/2} \\ \left(2t + 2\sin(t)\right)e^{1-\cos(t)} + \left(-5t - 5\sin(t)\right)e^{t^2/2} \end{bmatrix} \\
&= \begin{bmatrix} -e^{1-\cos(t)} + 3e^{-t^2/2} \\ 2e^{1-\cos(t)} - 5e^{-t^2/2} \end{bmatrix},
\end{aligned}$$

which is the same result as before.

4. A control system is described in state-space representation, such that

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t),$$

where $\boldsymbol{u}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ are the input, the state, and the output variables, respectively. For the following A, B, C, and D matrices, determine whether the system is asymptotically stable, marginally stable, or unstable; and whether it is bounded-input-bounded-output stable or not. Justify your answer.

(a)

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and } D = 0.$$

Solution: In order to determine the stability of the system, we first need to determine its eigenvalues and poles. Since in this case, the state matrix A is in modal form, we observe the eigenvalues directly from the block-diagonal matrices

$$A_1 = A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

From

$$\det(sI - A_1) = \det(sI - A_2) = \det\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} = s^2 + 1,$$

we get $\lambda_{1,2} = \lambda_{3,4} = \pm j$. The eigenvalues are on the imaginary axis and they are repeated. If they are cascaded, a sinusoidal response generated by the first set would result in a ramp envelope by the second one. In this case, since the state matrix A is block-diagonal, the two imaginary-valued eigenvalues don't affect each other, or they are not cascaded. Therefore, each set of the eigenvalues $\lambda_{1,2}$ and $\lambda_{3,4}$ would generate a sinusoidal response resulting in a marginally stable response. Since there are no more eigenvalues, we conclude that the system is marginally stable.

In order to determine the bounded-input-bounded-output stability of the system, we may determine the transfer function and observe the poles of the system after all the reductions. The transfer function of the system is given by

$$\mathcal{L}\left[\boldsymbol{y}\right](s) = \big(C(sI-A)^{-1}B + D\big)\,\mathcal{L}\left[\boldsymbol{u}\right](s)$$

where $\mathcal{L}\left[(\cdot)\right](s)$ is the laplace transform, and I is the appropriately dimensioned identity matrix. In our case,

$$\begin{aligned}
&C(sI-A)^{-1}B + D \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & 1 & 0 & 0 \\ -1 & s & 0 & 0 \\ 0 & 0 & s & 1 \\ 0 & 0 & -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \left(\frac{1}{(s^2+1)^2} \begin{bmatrix} s(s^2+1) & -(s^2+1) & 0 & 0 \\ s^2+1 & s(s^2+1) & 0 & 0 \\ 0 & 0 & s(s^2+1) & -(s^2+1) \\ 0 & 0 & s^2+1 & s(s^2+1) \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s/(s^2+1) & -1/(s^2+1) & 0 & 0 \\ 1/(s^2+1) & s/(s^2+1) & 0 & 0 \\ 0 & 0 & s/(s^2+1) & -1/(s^2+1) \\ 0 & 0 & 1/(s^2+1) & s/(s^2+1) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \frac{s}{s^2+1}.
\end{aligned}$$

We realize that a complex pair on the imaginary axis are visible in the transfer function. Since the complex pair on the imaginary axis would generate a sinusoidal response with a ramp envelope for a sinusoidal input of the same frequency, the system is not bounded-input-bounded-output stable.

In summary, the system is marginally stable, and it is not bounded-input-bounded-output stable.

(b)

$$A = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and } D = 0.$$

Solution: In this case, the state matrix A is not completely in modal form, but it is in upper-block-triangular form. As a result, we still can observe the eigenvalues directly from the block-matrices on the diagonal

$$A_1 = A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

From

$$\det(sI - A_1) = \det(sI - A_2) = \det \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} = s^2 + 1,$$

we get $\lambda_{1,2} = \lambda_{3,4} = \pm j$. The eigenvalues are on the imaginary axis and they are repeated. If they are cascaded, a sinusoidal response generated by the first set would result in a ramp envelope by the second one. To determine whether the system can be diagonalized or will be in jordan form, we need to check the rank condition.

$$\operatorname{rank}(A - \lambda I)_{\lambda=j} = \operatorname{rank} \begin{bmatrix} -\lambda & -1 & 1 & 0 \\ 1 & -\lambda & 0 & -1 \\ 0 & 0 & -\lambda & -1 \\ 0 & 0 & 1 & -\lambda \end{bmatrix}_{\lambda=j}$$

$$= \operatorname{rank} \begin{bmatrix} -j & -1 & 1 & 0 \\ 1 & -j & 0 & -1 \\ 0 & 0 & -j & -1 \\ 0 & 0 & 1 & -j \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -j & -1 & 1 & 0 \\ j & 1 & 0 & -j \\ 0 & 0 & 1 & -j \\ 0 & 0 & 1 & -j \end{bmatrix}$$

$$= \operatorname{rank} \begin{bmatrix} -j & -1 & 1 & 0 \\ 0 & 0 & 1 & -j \\ 0 & 0 & 1 & -j \\ 0 & 0 & 1 & -j \end{bmatrix} = 2.$$

The rank drop is two for the twice-repeated eigenvalue $\lambda_{1,3} = j$. As a result, we can obtain two linearly independent eigenvectors for this eigenvalue, and the system can be diagonalized. The eigenvalues on the imaginary axis are not cascaded, and the system is marginally stable.

In order to determine the bounded-input-bounded-output stability of the system, we may determine the transfer function and observe the poles of the system after all the reductions. The transfer function of the system is given by

$$\mathcal{L}[\boldsymbol{y}](s) = \left(C(sI - A)^{-1}B + D\right)\mathcal{L}[\boldsymbol{u}](s)$$

where $\mathcal{L}[(\cdot)](s)$ is the laplace transform, and I is the appropriately dimensioned identity matrix.

In our case,

$$
\begin{aligned}
&C(sI-A)^{-1}B+D \\
&= \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{cccc} s & 1 & 0 & 0 \\ -1 & s & 0 & -1 \\ 0 & 0 & s & 1 \\ 0 & 0 & -1 & s \end{array}\right]^{-1} \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right] + 0 \\
&= \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right] \left(\frac{1}{(s^2+1)^2} \left[\begin{array}{cccc} s(s^2+1) & -(s^2+1) & s^2+1 & 0 \\ s^2+1 & s(s^2+1) & 0 & -(s^2+1) \\ 0 & 0 & s(s^2+1) & -(s^2+1) \\ 0 & 0 & s^2+1 & s(s^2+1) \end{array}\right] \right) \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right] \\
&= \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{cccc} s/(s^2+1) & -1/(s^2+1) & 1/(s^2+1) & 0 \\ 1/(s^2+1) & s/(s^2+1) & 0 & -1/(s^2+1) \\ 0 & 0 & s/(s^2+1) & -1/(s^2+1) \\ 0 & 0 & 1/(s^2+1) & s/(s^2+1) \end{array}\right] \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right] \\
&= \frac{s}{s^2+1}.
\end{aligned}
$$

We realize that a complex pair on the imaginary axis are visible in the transfer function. Since the complex pair on the imaginary axis would generate a sinusoidal response with a ramp envelope for a sinusoidal input of the same frequency, the system is not bounded-input-bounded-output stable.

In summary, the system is marginally stable, and it is not bounded-input-bounded-output stable.

(c)

$$
A = \left[\begin{array}{cccc} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right], \qquad B = \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right], \qquad C = \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \end{array}\right], \text{ and } D = 0.
$$

Solution: In this case, the state matrix A is not completely in modal form, but it is in upper-block-triangular form. As a result, we still can observe the eigenvalues directly from the block-matrices on the diagonal

$$
A_1 = A_2 = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right].
$$

From

$$
\det(sI - A_1) = \det(sI - A_2) = \det \left[\begin{array}{cc} s & 1 \\ -1 & s \end{array}\right] = s^2 + 1,
$$

we get $\lambda_{1,2} = \lambda_{3,4} = \pm j$. The eigenvalues are on the imaginary axis and they are repeated. If they are cascaded, a sinusoidal response generated by the first set would result in a ramp envelope by the second one. To determine whether the system can be diagonalized or will be in jordan form, we need to check the rank condition.

$$\operatorname{rank}\big(A-\lambda I\big)_{\lambda=j} = \operatorname{rank}\begin{bmatrix} -\lambda & -1 & 0 & 0 \\ 1 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & -1 \\ 0 & 0 & 1 & -\lambda \end{bmatrix}_{\lambda=j}$$

$$= \operatorname{rank}\begin{bmatrix} -j & -1 & 0 & 0 \\ 1 & -j & 1 & 0 \\ 0 & 0 & -j & -1 \\ 0 & 0 & 1 & -j \end{bmatrix} = \operatorname{rank}\begin{bmatrix} -j & -1 & 0 & 0 \\ j & 1 & j & 0 \\ 0 & 0 & -j & -1 \\ 0 & 0 & -j & -1 \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} -j & -1 & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & -j & -1 \\ 0 & 0 & -j & -1 \end{bmatrix} = 3.$$

The rank drop is only one for the repeated eigenvalue $\lambda_{1,3} = j$. As a result, we can only obtain one eigenvector for this eigenvalue, and the system cannot be diagonalized. In jordan form, the eigenvalues on the imaginary axis are cascaded, and the system is unstable.

In order to determine the bounded-input-bounded-output stability of the system, we may determine the transfer function and observe the poles of the system after all the reductions. The transfer function of the system is given by

$$\mathcal{L}\left[\boldsymbol{y}\right](s) = \big(C(sI-A)^{-1}B + D\big)\,\mathcal{L}\left[\boldsymbol{u}\right](s)$$

where $\mathcal{L}\left[\cdot\right](s)$ is the laplace transform, and I is the appropriately dimensioned identity matrix. In our case,

$$C(sI-A)^{-1}B + D$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s & 1 & 0 & 0 \\ -1 & s & 0 & -1 \\ 0 & 0 & s & 1 \\ 0 & 0 & -1 & s \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}\left(\frac{1}{(s^2+1)^2}\begin{bmatrix} s(s^2+1) & -(s^2+1) & -s & 1 \\ s^2+1 & s(s^2+1) & s^2 & -s \\ 0 & 0 & s(s^2+1) & -(s^2+1) \\ 0 & 0 & s^2+1 & s(s^2+1) \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s/(s^2+1) & -1/(s^2+1) & -s/(s^2+1)^2 & 1/(s^2+1)^2 \\ 1/(s^2+1) & s/(s^2+1) & s^2/(s^2+1)^2 & -s/(s^2+1)^2 \\ 0 & 0 & s/(s^2+1) & -1/(s^2+1) \\ 0 & 0 & 1/(s^2+1) & s/(s^2+1) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= 0.$$

The transfer function is zero, and the system is bounded-input-bounded-output stable.

In summary, the system is unstable, and it is bounded-input-bounded-output stable.