EE 431 | Exam#2 | Dec. 05, 2006

75 minutes

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1. A control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -4 & 0 & -29 & -7 \\ 1 & -1 & 8 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & -4 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} -5 \\ 2 \\ 0 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 2 & 6 & 7 & 4 \end{bmatrix} \boldsymbol{x}(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively. Obtain its kalman decomposition that separates the controllable, uncontrollable, observable, and unobservable portions. Clearly mark the portions on the decomposed system. (40pts)

2. The dynamics of a control system are described by

$$\boldsymbol{Y}(s) = \begin{bmatrix} \dfrac{s+3}{s(s+1)^2} & \dfrac{s+2}{s^2(s+1)^2} \\ \dfrac{4}{s^2} & \dfrac{2}{s(s+1)} \end{bmatrix} \boldsymbol{U}(s),$$

where $\boldsymbol{U}$ and $\boldsymbol{Y}$ are the laplace transforms of the input and the output vectors, respectively. Obtain a minimal state-space representation of the system. Show all your work. (40pts)

3. A continuous-time linear control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \boldsymbol{x}(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively. Design a feedback controller for the system, such that the 2% settling time is less than 2 seconds, and the output response is critically-damped. (20pts)

HINT: The 2% settling time $t_{2\%_s} = (4/\sigma_o)$, and the maximum percent-overshoot

$$M_{p\%} = e^{-\left(\zeta/\sqrt{1-\zeta^2}\right)\pi}\, 100\%$$

for a second-order system with no zero and the poles at $-\sigma_o \pm j\omega_d = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\omega_n$.

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1. A control system is described by

$$\dot{x}(t) = \begin{bmatrix} -4 & 0 & -29 & -7 \\ 1 & -1 & 8 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & -4 \end{bmatrix} x(t) + \begin{bmatrix} -5 \\ 2 \\ 0 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 2 & 6 & 7 & 4 \end{bmatrix} x(t),$$

where u, x, and y are the input, the state, and the output variables, respectively. Obtain its kalman decomposition that separates the controllable, uncontrollable, observable, and unobservable portions. Clearly mark the portions on the decomposed system.

Solution: The kalman decomposition will transform the system, such that

$$\begin{bmatrix} \dot{x}_{c,o}(t) \\ \dot{x}_{c,\bar{o}}(t) \\ \dot{x}_{\bar{c},o}(t) \\ \dot{x}_{\bar{c},\bar{o}}(t) \end{bmatrix} = \begin{bmatrix} A_{c,o} & 0 & * & 0 \\ * & A_{c,\bar{o}} & * & * \\ 0 & 0 & A_{\bar{c},o} & 0 \\ 0 & 0 & * & A_{\bar{c},\bar{o}} \end{bmatrix} \begin{bmatrix} x_{c,o}(t) \\ x_{c,\bar{o}}(t) \\ x_{\bar{c},o}(t) \\ x_{\bar{c},\bar{o}}(t) \end{bmatrix} + \begin{bmatrix} B_{c,o} \\ B_{c,\bar{o}} \\ 0 \\ 0 \end{bmatrix} u(t),$$

and

$$y(t) = \begin{bmatrix} C_{c,o} & 0 & C_{\bar{o},c} & 0 \end{bmatrix} \begin{bmatrix} x_{c,o}(t) \\ x_{c,\bar{o}}(t) \\ x_{\bar{c},o}(t) \\ x_{\bar{c},\bar{o}}(t) \end{bmatrix} + Du(t),$$

where the controllable, uncontrollable, observable, and unobservable portions are denoted by the subscripts c, $\bar{c}$, o, and $\bar{o}$, respectively. To separate the controllable and the uncontrollable portions, we need to pick the linearly independent column vectors from the controllability matrix. In an nth order system that is described by

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

where u, x, and y are the input, the state, and the output variables, respectively; the controllability matrix is given by

$$\mathcal{C}(A,B) = \begin{bmatrix} B \mid AB \mid \cdots \mid A^{n-1}B \end{bmatrix}.$$

In our case, the system order $n = 4$, and

$$\mathcal{C}(A,B) = \begin{bmatrix} B \mid AB \mid A^2B \mid A^3B \end{bmatrix} = \begin{bmatrix} -5 & 20 & -80 & 320 \\ 2 & -7 & 27 & -170 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the controllability matrix, we observe that there are only two linearly independent columns:

$$\begin{bmatrix} 5 & 20 \\ 2 & -7 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

After supplying two more linearly independent columns to the above columns, the transformation matrix could be

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since the transformation matrix could be the identity matrix, we conclude that the controllable and uncontrollable portions are already separated. The first two states are controllable, and the last two states are not controllable.

To separate the observable and the unobservable portions, we need to pick the linearly independent row vectors from the observability matrix

$$\mathcal{O}(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

In our case, the observability matrix is

$$\mathcal{O}(C, A) = \begin{bmatrix} 2 & 6 & 7 & 4 \\ -2 & -6 & -22 & -24 \\ 2 & 6 & 82 & 104 \\ -2 & -6 & -322 & -424 \end{bmatrix}.$$

To separate the observable and the unobservable portions, we need to pick the linearly independent row vectors from the observability matrix. There are only two linearly independent row vectors in the observability matrix:

$$\begin{bmatrix} 2 & 6 & 7 & 4 \\ -2 & -6 & -22 & -24 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 & 6 & 7 & 4 \\ 0 & 0 & -15 & -20 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix}.$$

The rest of the rows need to be supplied with other vectors that are linearly independent to the original vector in the transformation matrix. However, we need to determine where to place these two rows in a four-row matrix.

If we consider only the controllable portion, we get

$$A_c = \begin{bmatrix} -4 & 0 \\ 1 & -1 \end{bmatrix}, \qquad B_c = \begin{bmatrix} -5 \\ 2 \end{bmatrix},$$
$$C_c = \begin{bmatrix} 2 & 6 \end{bmatrix}.$$

The observability matrix for the controllable portion is

$$\mathcal{O}(C_c, A_c) = \begin{bmatrix} 2 & 6 \\ -2 & -6 \end{bmatrix}.$$

In other words, only one of the states in the controllable portion is observable, and the corresponding row for that state starts with $\begin{bmatrix} 2 & 6 \end{bmatrix}$. Therefore, the transformation matrix

$$S^T = \begin{bmatrix} 1 & 3 & 2 & 0 \\ * & * & * & * \\ 0 & 0 & 3 & 4 \\ * & * & * & * \end{bmatrix}$$

puts the observable state in the controllable portion as the first state and the observable state in the uncontrollable portion as the third state. We fill the second and fourth rows with linearly independent vectors as long as the controllable and uncontrollable portions are not combined. One such transformation matrix is

$$S^T = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The new system matrices are

$$\bar{A} = S^T A (S^T)^{-1} = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 & -29 & -7 \\ 1 & -1 & 8 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & -4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & -4 & 2 & 7 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$\bar{B} = S^T B = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix},$$

and

$$\bar{C} = C(S^T)^{-1} = \begin{bmatrix} 2 & 6 & 7 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 & 1 & 0 \end{bmatrix}.$$

After marking the state variables, we get the kalman decomposition.

$$\begin{bmatrix} \dot{x}_{c,o}(t) \\ \hline \dot{x}_{c,\bar{o}}(t) \\ \hline \dot{x}_{\bar{c},o}(t) \\ \hline \dot{x}_{\bar{c},\bar{o}}(t) \end{bmatrix} = \left[\begin{array}{c|c|c|c} A_{c,o} & 0 & * & 0 \\ \hline * & A_{c,\bar{o}} & * & * \\ \hline 0 & 0 & A_{\bar{c},o} & 0 \\ \hline 0 & 0 & * & A_{\bar{c},\bar{o}} \end{array}\right] \begin{bmatrix} x_{c,o}(t) \\ \hline x_{c,\bar{o}}(t) \\ \hline x_{\bar{c},o}(t) \\ \hline x_{\bar{c},\bar{o}}(t) \end{bmatrix} + \begin{bmatrix} B_{c,o} \\ \hline B_{c,\bar{o}} \\ \hline 0 \\ \hline 0 \end{bmatrix} u(t)$$

$$= \left[\begin{array}{c|c|c|c} -1 & 0 & -1 & 0 \\ \hline 1 & -4 & 2 & 7 \\ \hline 0 & 0 & -4 & 0 \\ \hline 0 & 0 & -1 & 0 \end{array}\right] \begin{bmatrix} x_{c,o}(t) \\ \hline x_{c,\bar{o}}(t) \\ \hline x_{\bar{c},o}(t) \\ \hline x_{\bar{c},\bar{o}}(t) \end{bmatrix} + \begin{bmatrix} 1 \\ \hline 2 \\ \hline 0 \\ \hline 0 \end{bmatrix} u(t),$$

and

$$y(t) = \left[\begin{array}{c|c|c|c} C_{c,o} & 0 & C_{\bar{c},o} & 0 \end{array}\right] \begin{bmatrix} x_{c,o}(t) \\ \hline x_{c,\bar{o}}(t) \\ \hline x_{\bar{c},o}(t) \\ \hline x_{\bar{c},\bar{o}}(t) \end{bmatrix} = \left[\begin{array}{c|c|c|c} 2 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} x_{c,o}(t) \\ \hline x_{c,\bar{o}}(t) \\ \hline x_{\bar{c},o}(t) \\ \hline x_{\bar{c},\bar{o}}(t) \end{bmatrix}.$$

2. The dynamics of a control system are described by

$$\boldsymbol{Y}(s) = \begin{bmatrix} \dfrac{s+3}{s(s+1)^2} & \dfrac{s+2}{s^2(s+1)^2} \\ \dfrac{4}{s^2} & \dfrac{2}{s(s+1)} \end{bmatrix} \boldsymbol{U}(s),$$

where $\boldsymbol{U}$ and $\boldsymbol{Y}$ are the laplace transforms of the input and the output vectors, respectively. Obtain a minimal state-space representation of the system. Show all your work.

Solution: In order to have a state-space representation, we need to generate a coprime factorization of the system, such that

$$\boldsymbol{Y}(s) = D_L^{-1}(s) N_L(s) \boldsymbol{U}(s)$$

or

$$\boldsymbol{Y}(s) = N_R(s) D_R^{-1}(s) \boldsymbol{U}(s)$$

for some polynomial matrices N_L and D_L or N_R and D_R. The easiest approach to obtain an initial version of D is to multiply the common denominator by the identity matrix.

$$\boldsymbol{Y}(s) = \begin{bmatrix} \frac{1}{s^2(s+1)^2} & 0 \\ 0 & \frac{1}{s^2(s+1)^2} \end{bmatrix} \begin{bmatrix} s(s+3) & s+2 \\ 4(s+1)^2 & 2s(s+1) \end{bmatrix} \boldsymbol{U}(s)$$
$$= \begin{bmatrix} s^2(s+1)^2 & 0 \\ 0 & s^2(s+1)^2 \end{bmatrix}^{-1} \begin{bmatrix} s(s+3) & s+2 \\ 4(s+1)^2 & 2s(s+1) \end{bmatrix} \boldsymbol{U}(s).$$

The above equation is in the initial left-factorization form, where $\boldsymbol{Y}(s) = D_0^{-1}(s)N_0^{-1}(s)\boldsymbol{U}(s)$. To obtain a left coprime factorization, we form an augmented matrix from N_0 and D_0, and perform row operations until we obtain a reduced form.

$$\left[\, N_0(s) \,\middle|\, D_0(s) \,\right] = \left[\begin{array}{cc|cc} s(s+3) & s+2 & s^2(s+1)^2 & 0 \\ 4(s+1)^2 & 2s(s+1) & 0 & s^2(s+1)^2 \end{array}\right].$$

Dividing the second row by $(s+1)$, we get

$$\left[\, N_1(s) \,\middle|\, D_1(s) \,\right] = \left[\begin{array}{cc|cc} s(s+3) & s+2 & s^2(s+1)^2 & 0 \\ 4(s+1) & 2s & 0 & s^2(s+1) \end{array}\right].$$

We can observe that the factorization is already in a coprime factorization, since the rank of the augmented matrix will not drop for any value of s, and the degree of the determinant of $D(s)$ is the same as the sum of its highest row degrees. As a result, all we need to do is to realize the left coprime factorization. First, we decompose $D(s) = D_1(s)$ and $N(s) = N_1(s)$, such that

$$D(s) = S_r(s)D_{h_r} + \Psi_r(s)D_{l_r},$$

and

$$N(s) = S_r(s)N_{h_r} + \Psi_r(s)N_{l_r},$$

where

$$S_r(s) = \left[\begin{array}{c|c|c} s^{l_1} & & \\ \hline & s^{l_2} & \\ \hline & & \ddots \end{array}\right] \text{ and } \Psi_r(s) = \left[\begin{array}{ccc|ccc|c} s^{l_1-1} & \dots & 1 & & & & \\ \hline & & & s^{l_2-1} & \dots & 1 & \\ \hline & & & & & & \ddots \end{array}\right]$$

are block-diagonal matrices, and l_i is the highest degree of the polynomials on the ith row of $D(s)$. In our case, $l_1 = 4$, $l_2 = 3$,

$$S_r(s) = \left[\begin{array}{c|c} s^4 & 0 \\ \hline 0 & s^3 \end{array}\right], \text{ and } \Psi_r(s) = \left[\begin{array}{cccc|ccc} s^3 & s^2 & s & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & s^2 & s & 1 \end{array}\right].$$

The decompositions become

$$D(s) = \begin{bmatrix} s^4+2s^3+s^2 & 0 \\ 0 & s^4+2s^3+s^2 \end{bmatrix} = S_r(s)D_{h_r} + \Psi_r(s)D_{l_r}$$

$$= \left[\begin{array}{c|c} s^4 & 0 \\ \hline 0 & s^3 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right] + \left[\begin{array}{cccc|ccc} s^3 & s^2 & s & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & s^2 & s & 1 \end{array}\right] \left[\begin{array}{c|c} 2 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}\right],$$

and

$$N(s) = \begin{bmatrix} s^2+3s & s+2 \\ 4s+4 & 2s \end{bmatrix} = S_r(s)N_{h_r} + \Psi_r(s)N_{l_r}$$

$$= \left[\begin{array}{c|c} s^4 & 0 \\ \hline 0 & s^3 \end{array}\right] \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{cccc|ccc} s^3 & s^2 & s & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & s^2 & s & 1 \end{array}\right] \left[\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ 3 & 1 \\ 0 & 2 \\ \hline 0 & 0 \\ 4 & 2 \\ 4 & 0 \end{array}\right].$$

The controller canonical-form realization is, then, given by

$$\dot{\boldsymbol{x}}(t) = \left(A_o^0 - B_o^0 D_{l_r} D_{h_r}^{-1} C_o^0\right) \boldsymbol{x}(t) + B_o^0 \left(N_{l_r} - D_{l_r} D_{h_r}^{-1} N_{h_r}\right) \boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = \left(D_{h_r}^{-1} C_o^0\right) \boldsymbol{x}(t) + \left(D_{h_r}^{-1} N_{h_r}\right) \boldsymbol{u}(t),$$

where

$$A_o^0 = \left[\begin{array}{cccc|cccc|c} 0 & 1 & \dots & 0 & & & & & \\ \vdots & \ddots & \ddots & \vdots & & & & & \\ 0 & 0 & \dots & 1 & & & & & \\ 0 & 0 & \dots & 0 & & & & & \\ \hline & & & & 0 & 1 & \dots & 0 & \\ & & & & \vdots & \ddots & \ddots & \vdots & \\ & & & & 0 & 0 & \dots & 1 & \\ & & & & 0 & 0 & \dots & 0 & \\ \hline & & & & & & & & \ddots \end{array}\right],$$

B_o^0 is the identity matrix with dimension $\sum_i l_i$, and

$$C_o^0 = \left[\begin{array}{cccc|cccc|c} 1 & 0 & \dots & 0 & & & & & \\ \hline & & & & 1 & 0 & \dots & 0 & \\ \hline & & & & & & & & \ddots \end{array}\right].$$

In our case,

$$A_o^0 = \left[\begin{array}{cccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

B_o^0 is the 7 dimensional identity matrix, and

$$C_o^0 = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right].$$

So,

$$A_o = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_o = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 3 & 1 \\ 0 & 2 \\ 0 & 0 \\ 4 & 2 \\ 4 & 0 \end{bmatrix},$$

and

$$C_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Therefore, one possible state-space representation of the system is given by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 3 & 1 \\ 0 & 2 \\ 0 & 0 \\ 4 & 2 \\ 4 & 0 \end{bmatrix} \boldsymbol{u}(t),$$

$$\boldsymbol{y}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \boldsymbol{x}(t),$$

where $\boldsymbol{u}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ are the input, the state, and the output variables, respectively.

3. A continuous-time linear control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \boldsymbol{x}(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively. Design a feedback controller for the system, such that the 2% settling time is less than 2 seconds, and the output response is critically-damped.

HINT: The 2% settling time $t_{2\%_s} = (4/\sigma_o)$, and the maximum percent-overshoot

$$M_{p\%} = e^{-\left(\zeta/\sqrt{1-\zeta^2}\right)\pi}\,100\%$$

for a second-order system with no zero and the poles at $-\sigma_o \pm j\omega_d = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\omega_n$.

Solution: We determine the desired system closed-loop poles from the system requirements.

Given Requirements	General System Restrictions	Specific System Restrictions
2% settling-time for the unit-step input	$t_{2\%_s} \leq 2\,\text{s}$, or $\dfrac{4}{\sigma_o} \leq 2.$	$\sigma_o \geq 2$, since $t_{2\%_s} = 4/\sigma_o$.
The output response is critically-damped.	The dominant closed-loop poles are real and identical.	$\text{pole}_{1,2} = -\sigma_0$.

From the given requirements, we choose $\text{pole}_{1,2} = -\sigma_0 = -2.$. The desired characteristic polynomial q_{c_d} can be obtained from the desired-pole locations, where

$$q_{c_d}(s) = \big(s - (-2)\big)\big(s - (-2)\big) = s^2 + 4s + 4.$$

The characteristic polynomial q_c under state-feedback gain $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, such that the input $u = K\boldsymbol{x}$, can be determined from

$$\begin{aligned} q_c(s) &= \det\big(sI - (A + BK)\big) \\ &= \det\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right)\right) \\ &= s^2 + (-k_2 + 2)z + (-k_1 - 1). \end{aligned}$$

Setting $q_c(s) = q_{c_d}(s)$, we get

$$-k_1 - 1 = 4,$$

or $k_1 = -5$; and

$$-k_2 + 2 = 4,$$

or $k_2 = -2$. Therefore,

$$K = \begin{bmatrix} -5 & -2 \end{bmatrix},$$

or

$$u(t) = r(t) + \begin{bmatrix} -5 & -2 \end{bmatrix}\boldsymbol{x}(t) \text{ for } t \geq 0,$$

where r is the new reference input.