EE 431 | Exam#2 75 minutes | Nov. 29, 2007

1. Consider the autonomous linear control system

$$\dot{x}(t) = \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix} x(t),$$

where x is the state variable. Obtain a lyapunov function to prove its stability or instability. (25pts)

2. A continuous-time linear control system is described by

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t),$$

where u and x are the input and the state variables, respectively. Determine the control signal $u(t)$ for $0 \le t \le 1$, such that $x(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $x(1) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$. (25pts)

3. A continuous-time linear control system is described by

$$\dot{x}(t) = \begin{bmatrix} -2 & -2 \\ 2 & -7 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ -2 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 3 & -3 \end{bmatrix} x(k),$$

where u, x, and y are the input, the state, and the output variables, respectively. Determine whether or not an initial condition can be uniquely determined by observing the future values of the output and control. If such an observation is possible; then determine the initial condition $x(0)$, when the output is $y(t) = 1/2$, and $u(t) = 1$ for $t \ge 0$. (25pts)

4. A control system is described by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 3 \\ -1 & -2 & -2 \\ -1 & -1 & -3 \end{bmatrix} x(t) + \begin{bmatrix} -8 \\ 7 \\ 3 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 6 & 6 & 2 \end{bmatrix} x(t),$$

where u, x, and y are the input, the state, and the output variables, respectively. Obtain its kalman decomposition that separates the controllable, uncontrollable, observable, and unobservable portions. Clearly mark the portions on the decomposed system. (25pts)

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1. Consider the autonomous linear control system

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix} \boldsymbol{x}(t),$$

where $\boldsymbol{x}$ is the state variable. Obtain a lyapunov function to prove its stability or instability.

Solution: A lyapunov function for a linear system is $L(\boldsymbol{x}) = \boldsymbol{x}^T P \boldsymbol{x}$; such that if a positive definite and symmetric matrix P satisfies the lyapunov equation

$$A^T P + PA = -Q,$$

for the state matrix A and a positive definite and symmetric matrix Q, then the system is asymptotically stable. In our case

$$A = \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix}.$$

Since any positive definite and symmetric matrix Q should work, let Q be the 2×2 identity matrix, and

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}.$$

Checking for a solution to the lyapunov equation, we get

$$\begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix}^T \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

or

$$\begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Due to the symmetry, we get three equations from the matrix equation.

$$-6p_1 - 2p_2 = -1,$$

$$-p_1 - 6p_2 - p_3 = 0,$$

and

$$-2p_2 - 6p_3 = -1.$$

From the first and the third equations, we get $p_1 = (1 - 2p_2)/6$, and $p_3 = (1 - 2p_2)/6$. Substituting these expressions into the second equation, we get

$$-\left(\frac{1-2p_2}{6}\right) - 6p_2 - \left(\frac{1-2p_2}{6}\right) = 0,$$

or $p_2 = -1/16$. As a result, we get

$$P = (1/16) \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

From the positiveness of the principal minors of P, $p_1 = (3/16) > 0$ (or $p_3 = (3/16) > 0$), and $\det(P) = p_1 p_3 - p_2{}^2 = (1/32) > 0$; we conclude that P is positive definite. Since there exists a lyapunov function

$$L(\boldsymbol{x}) = (1/16)\, \boldsymbol{x}^T \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \boldsymbol{x} > 0,$$

such that

$$\frac{\mathrm{d}L(\boldsymbol{x})}{\mathrm{d}t} = -\boldsymbol{x}^T\boldsymbol{x} < 0 \text{ for all } \boldsymbol{x} \neq \mathbf{0};$$

the system is asymptotically stable.

2. A continuous-time linear control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t),$$

where u and $\boldsymbol{x}$ are the input and the state variables, respectively. Determine the control signal $u(t)$ for $0 \le t \le 1$, such that $\boldsymbol{x}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $\boldsymbol{x}(1) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

Solution: The general solution to the state-space representation of a linear system described by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

is obtained from

$$\boldsymbol{x}(t) = \Phi(t,0)\boldsymbol{x}(0) + \int_0^t \Phi(t,\tau)B\boldsymbol{u}(\tau)\,\mathrm{d}\tau,$$

where $\boldsymbol{u}$ and $\boldsymbol{x}$ are the input and the state variables, respectively, and

$$\Phi(t,t_0) = e^{A(t-t_0)}.$$

In our case, the state matrix A is in diagonal form, and

$$\Phi(t,t_0) = e^{A(t-t_0)} = \begin{bmatrix} e^{-(t-t_0)} & 0 \\ 0 & e^{-2(t-t_0)} \end{bmatrix}.$$

One method to solve the given control problem is to separate the control signal into two portions, such that one of the portions can be moved out of the integral, and the integral term is invertible. So, we let

$$u(\tau) = \left(\Phi(t,\tau)B\right)^T \boldsymbol{\xi}(t),$$

where $\boldsymbol{\xi}$ needs to be determined from the given conditions. With this control signal, we get

$$\begin{aligned} \boldsymbol{x}(t) &= \Phi(t,0)\boldsymbol{x}(0) + \left(\int_0^t \Phi(t,\tau)BB^T\Phi^T(t,\tau)\,\mathrm{d}\tau\right)\boldsymbol{\xi}(t) \\ &= \Phi(t,0)\boldsymbol{x}(0) + \mathfrak{C}(0,t)\boldsymbol{\xi}(t), \end{aligned}$$

where the controllability grammian

$$\mathfrak{C}(0,t) = \int_0^t \Phi(t,\tau)BB^T\Phi^T(t,\tau)\,\mathrm{d}\tau.$$

Since the controllability grammian always has an inverse, if the system is reachable; we get

$$\boldsymbol{\xi}(t) = \mathfrak{C}^{-1}(0,t)\Big(\boldsymbol{x}(t) - \Phi(t,0)\boldsymbol{x}(0)\Big),$$

or

$$u(\tau) = \big(\Phi(t,\tau)B\big)^T\boldsymbol{\xi}(t) = B^T\Phi^T(t,\tau)\mathfrak{C}^{-1}(0,t)\Big(\boldsymbol{x}(t) - \Phi(t,0)\boldsymbol{x}(0)\Big).$$

To determine the control signal, we compute

$$\begin{aligned}
\mathfrak{C}(0,1) &= \int_0^1 \Phi(1,\tau)BB^T\Phi^T(1,\tau)\,\mathrm{d}\tau \\
&= \int_0^1 \begin{bmatrix} e^{-(1-\tau)} & 0 \\ 0 & e^{-2(1-\tau)} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-(1-\tau)} & 0 \\ 0 & e^{-2(1-\tau)} \end{bmatrix}\mathrm{d}\tau \\
&= \int_0^1 \begin{bmatrix} e^{-2(1-\tau)} & e^{-3(1-\tau)} \\ e^{-3(1-\tau)} & e^{-4(1-\tau)} \end{bmatrix}\mathrm{d}\tau = \begin{bmatrix} (1/2)-(1/2)e^{-2} & (1/3)-(1/3)e^{-3} \\ (1/3)-(1/3)e^{-3} & (1/4)-(1/4)e^{-4} \end{bmatrix}.
\end{aligned}$$

As a result,

$$\begin{aligned}
u(\tau) &= B^T\Phi^T(1,\tau)\mathfrak{C}^{-1}(0,1)\Big(\boldsymbol{x}(1) - \Phi(1,0)\boldsymbol{x}(0)\Big) \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-(1-\tau)} & 0 \\ 0 & e^{-2(1-\tau)} \end{bmatrix}\begin{bmatrix} (1/2)-(1/2)e^{-2} & (1/3)-(1/3)e^{-3} \\ (1/3)-(1/3)e^{-3} & (1/4)-(1/4)e^{-4} \end{bmatrix}^{-1} \\
&\qquad\qquad \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} e^{-1} & 0 \\ 0 & e^{-2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\
&= -\left(\frac{e^{-1}}{(1-e^{-1})^4(1+4e^{-1}+e^{-2})/72}\right)\left(\Big((1-e^{-4})/4\Big)e^{-(1-\tau)} - \Big((1-e^{-3})/3\Big)e^{-2(1-\tau)}\right) \\
&= -15.6183e^{-(1-\tau)} + 20.1568e^{-2(1-\tau)},
\end{aligned}$$

or

$$u(t) = -15.6183e^{-(1-t)} + 20.1568e^{-2(1-t)} \text{ for } 0 \le t \le 1.$$

The above solution generates a continuous-time control function. An alternative, but inferior and unrealizable, solution involves a linear combination of dirac-delta function (distribution) and its derivatives. If the system is reachable, there exists such a linear combination. For a second order system, we only need two terms in this linear combination. So, we let

$$u(t) = u_0\delta(t) + u_1\dot{\delta}(t).$$

Substituting the input to the solution equation for the state variable, we get

$$\begin{aligned}
\boldsymbol{x}(t) &= e^{At}\boldsymbol{x}(0_-) + \int_{0_-}^{t} e^{A(t-\tau)}B\left(u_0\delta(\tau) + u_1\dot{\delta}(\tau)\right)\,\mathrm{d}\tau \\
&= e^{At}\boldsymbol{x}(0_-) + \left(e^{A(t-\tau)}Bu_0 + (-1)\frac{\mathrm{d}}{\mathrm{d}\tau}\left(e^{A(t-\tau)}Bu_1\right)\right)_{\tau=0} \\
&= e^{At}\boldsymbol{x}(0_-) + \left(e^{At}Bu_0 + e^{At}ABu_1\right) \\
&= e^{At}\boldsymbol{x}(0_-) + e^{At}\begin{bmatrix} B & AB \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix};
\end{aligned}$$

or

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \mathcal{C}(A,B)^{-1}(e^{-At}\boldsymbol{x}(t) - \boldsymbol{x}(0_-)),$$

where $\mathcal{C}(A,B)$ is the controllability matrix. In our case, the system order $n = 2$, and

$$\mathcal{C}(A,B) = \left[\; B \mid AB \mid \cdots \mid A^{n-1}B \;\right] = \left[\; B \mid AB \;\right] = \left[\begin{array}{c|c} 1 & -1 \\ 1 & -2 \end{array}\right].$$

For $t = 1$, we get

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}^{-1} \left(e^{-A}\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

Therefore,

$$u(t) = -2\delta(t) - \dot{\delta}(t)$$

is an alternative solution.

3. A continuous-time linear control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -2 & -2 \\ 2 & -7 \end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix} -1 \\ -2 \end{bmatrix}u(t),$$
$$y(t) = \begin{bmatrix} 3 & -3 \end{bmatrix}\boldsymbol{x}(k),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively. Determine whether or not an initial condition can be uniquely determined by observing the future values of the output and control. If such an observation is possible; then determine the initial condition $\boldsymbol{x}(0)$, when the output is $y(t) = 1/2$, and $u(t) = 1$ for $t \geq 0$.

Solution: The property of determining the initial condition from the future values of the output is the observability property. To ensure observability of the system, the rank of the observability matrix should be full. The observability matrix for an nth order system

$$\mathcal{O}(C,A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix},$$

where A and C are the state and the output matrices of the system, respectively. In our case,

$$\mathcal{O}(C,A) = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -12 & 15 \end{bmatrix}.$$

In our system, we need to have 2 linearly independent rows or columns of $\mathcal{O}$ for reachability. Since the determinant of $\mathcal{O}$ is non-zero, we conclude that the system is observable and the initial state variable can be obtained from the output variables. To determine the initial conditions, we may use a formula that can be derived by considering repeated derivatives of the output equation.

$$\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{y}^{(1)} \\ \vdots \\ \boldsymbol{y}^{(n-1)} \end{bmatrix} = \mathcal{O}(C,A)\boldsymbol{x}(0) + \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ C^{n-1}B & C^{n-2}B & \cdots & D \end{bmatrix}\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{u}^{(1)} \\ \vdots \\ \boldsymbol{u}^{(n-1)} \end{bmatrix}.$$

In our case, $n = 2$ and $t = 0$,

$$\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -12 & 15 \end{bmatrix} \boldsymbol{x}(0) + \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solving for the unknown initial condition, we get

$$\boldsymbol{x}(0) = \begin{bmatrix} 3 & -3 \\ -12 & 15 \end{bmatrix}^{-1} \left(\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right);$$

and after some matrix manipulations, we obtain

$$\boldsymbol{x}(0) = \begin{bmatrix} -1/6 \\ -1/3 \end{bmatrix}.$$

4. A control system is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 & 3 \\ -1 & -2 & -2 \\ -1 & -1 & -3 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} -8 \\ 7 \\ 3 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 6 & 6 & 2 \end{bmatrix} \boldsymbol{x}(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively. Obtain its kalman decomposition that separates the controllable, uncontrollable, observable, and unobservable portions. Clearly mark the portions on the decomposed system.

Solution: The kalman decomposition will transform the system, such that

$$\begin{bmatrix} \dot{\boldsymbol{x}}_{\mathrm{c,o}}(t) \\ \dot{\boldsymbol{x}}_{\mathrm{c,\bar{o}}}(t) \\ \dot{\boldsymbol{x}}_{\mathrm{\bar{c},o}}(t) \\ \dot{\boldsymbol{x}}_{\mathrm{\bar{c},\bar{o}}}(t) \end{bmatrix} = \begin{bmatrix} A_{\mathrm{c,o}} & 0 & * & 0 \\ * & A_{\mathrm{c,\bar{o}}} & * & * \\ 0 & 0 & A_{\mathrm{\bar{c},o}} & 0 \\ 0 & 0 & * & A_{\mathrm{\bar{c},\bar{o}}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{\mathrm{c,o}}(t) \\ \boldsymbol{x}_{\mathrm{c,\bar{o}}}(t) \\ \boldsymbol{x}_{\mathrm{\bar{c},o}}(t) \\ \boldsymbol{x}_{\mathrm{\bar{c},\bar{o}}}(t) \end{bmatrix} + \begin{bmatrix} B_{\mathrm{c,o}} \\ B_{\mathrm{c,\bar{o}}} \\ 0 \\ 0 \end{bmatrix} \boldsymbol{u}(t),$$

and

$$\boldsymbol{y}(t) = \begin{bmatrix} C_{\mathrm{c,o}} & 0 & C_{\mathrm{\bar{o},c}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{\mathrm{c,o}}(t) \\ \boldsymbol{x}_{\mathrm{c,\bar{o}}}(t) \\ \boldsymbol{x}_{\mathrm{\bar{c},o}}(t) \\ \boldsymbol{x}_{\mathrm{\bar{c},\bar{o}}}(t) \end{bmatrix} + D\boldsymbol{u}(t),$$

where the controllable, uncontrollable, observable, and unobservable portions are denoted by the subscripts c, $\bar{\mathrm{c}}$, o, and $\bar{\mathrm{o}}$, respectively.

To separate the controllable and the uncontrollable portions, we need to pick the linearly independent column vectors from the controllability matrix. In an nth order system that is described by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$

$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t),$$

where $\boldsymbol{u}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ are the input, the state, and the output variables, respectively; the controllability matrix is given by

$$\mathcal{C}(A, B) = \begin{bmatrix} B \mid AB \mid \cdots \mid A^{n-1}B \end{bmatrix}.$$

In our case, the system order $n = 3$, and

$$\mathcal{C}(A,B) = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} -8 & 16 & -36 \\ 7 & -12 & 24 \\ 3 & -8 & 20 \end{bmatrix}.$$

From the controllability matrix, we observe that there are three linearly independent columns and the matrix is invertible. Therefore, all the states are controllable.

To separate the observable and the unobservable portions, we need to pick the linearly independent row vectors from the observability matrix

$$\mathcal{O}(C,A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

In our case, the observability matrix is

$$\mathcal{O}(C,A) = \begin{bmatrix} 6 & 6 & 2 \\ -8 & -8 & 0 \\ 8 & 8 & -8 \end{bmatrix}.$$

To separate the observable and the unobservable portions, we need to pick the linearly independent row vectors from the observability matrix. There are only two linearly independent row vectors in the observability matrix:

$$\begin{bmatrix} -8 & -8 & 0 \\ 8 & 8 & -8 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The remaining row needs to be supplied with another vector that is linearly independent to the original vectors in the transformation matrix. With one such choice of $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, the transformation matrix

$$S^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

puts the observable portion as the first two states and the unobservable portion as the third state. The new system matrices are

$$\bar{A} = S^T A (S^T)^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ -1 & -2 & -2 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ -1 & -3 & 0 \\ 1 & 3 & -1 \end{bmatrix},$$

$$\bar{B} = S^T B = \begin{bmatrix} 0 & 1 & 3 \\ -1 & -2 & -2 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} -8 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -8 \end{bmatrix},$$

and

$$\bar{C} = C(S^T)^{-1} = \begin{bmatrix} 6 & 6 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ -1 & -2 & -2 \\ -1 & -1 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} 6 & 2 & 0 \end{bmatrix}.$$

After marking the state variables, we get the kalman decomposition when all states are controllable.

$$\left[\begin{array}{c} \dot{x}_{o_1}(t) \\ \dot{x}_{o_2}(t) \\ \hline \dot{x}_{\bar{o}}(t) \end{array}\right] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & -3 & 0 \\ \hline 1 & 3 & -1 \end{array}\right] \left[\begin{array}{c} x_{o_1}(t) \\ x_{o_2}(t) \\ \hline x_{\bar{o}}(t) \end{array}\right] + \left[\begin{array}{c} -1 \\ 3 \\ \hline -8 \end{array}\right] u(t),$$

and

$$y(t) = \left[\begin{array}{cc|c} 6 & 2 & 0 \end{array}\right] \left[\begin{array}{c} x_{o_1}(t) \\ x_{o_2}(t) \\ \hline x_{\bar{o}}(t) \end{array}\right].$$