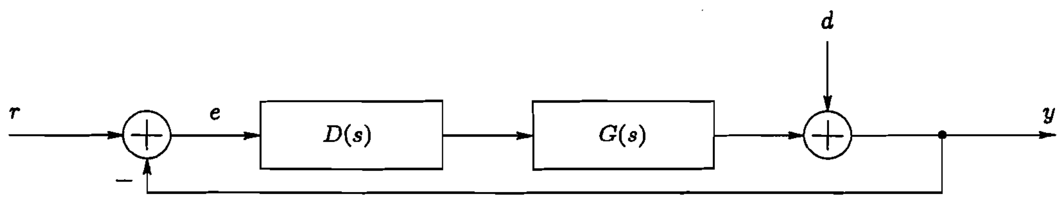
EE 231 | Exam#2 | Nov. 29, 2007

75 minutes

Copyright © 2007 by Levent Acar. All rights reserved. No parts of this document may be reproduced, stored in a retrieval system, or transmitted in any form or by any means without the written permission of the copyright holder(s).

1. Consider the following control system with the reference input r and the disturbance signal d.

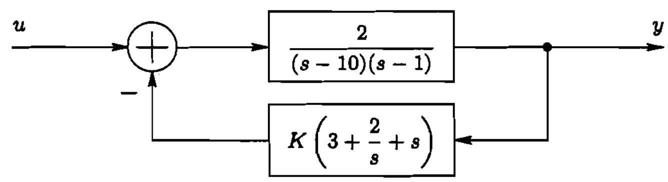


For the case when

$$G(s) = \frac{1}{s},$$

design a minimal-order controller, such that the output follows an alternate-current (AC) reference-input with a frequency of 4 rad/s as well as a step reference-input; and it also rejects a direct-current (DC) disturbance signal. (25pts)

2. Consider the following control system.



Determine the range of the constant K, such that the system is asymptotically stable. (25pts)

3. Consider the negative-feedback control-system with the following open-loop transfer-function. Construct the root-locus diagram. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angle of arrivals and/or departures.

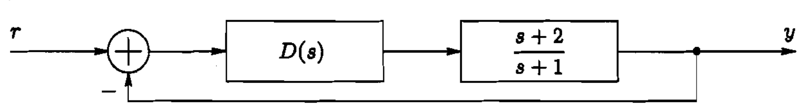
$$G(s) = K\frac{s+4}{(s+1)^3}.$$

(25pts)

4. Sketch the location of the closed-loop poles for the following feedback control system under a proportional-integral controller

$$D(s) = 1 + \frac{K}{s}$$

for $K > 0$.

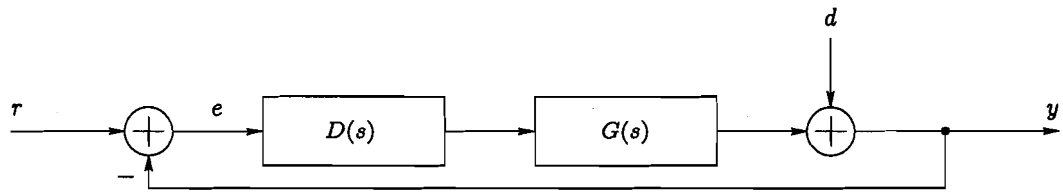


Show all the important features. (25pts)

EE 231 | **Exam#2 Solutions** | **Nov. 29, 2007**

Copyright © 2007 by Levent Acar. All rights reserved. No parts of this document may be reproduced, stored in a retrieval system, or transmitted in any form or by any means without the written permission of the copyright holder(s).

1. Consider the following control system with the reference input r and the disturbance signal d.



For the case when

$$G(s) = \frac{1}{s},$$

design a minimal-order controller, such that the output follows an alternate-current (AC) reference-input with a frequency of 4 rad/s as well as a step reference-input; and it also rejects a direct-current (DC) disturbance signal.

Solution: In order to follow any reference input and to reject a disturbance signal at the output, we need to match the non-asymptotically stable poles of the input and the disturbance in the open-loop gain of the system. In the case of the reference input, we need to have poles at $s = \pm j4$ for the AC signal and $s = 0$ for the step signal. To reject a DC disturbance, we need to match the disturbance pole at $s = 0$. However since $G(s)$ already has a pole at $s = 0$, that portion of the requirement is already satisfied. With these choices, the open-loop gain

$$D(s)G(s) = \left(\frac{1}{s^2+4^2}\,D'(s)\right)\left(\frac{1}{s}\right) = \frac{1}{s(s^2+16)}D'(s),$$

where

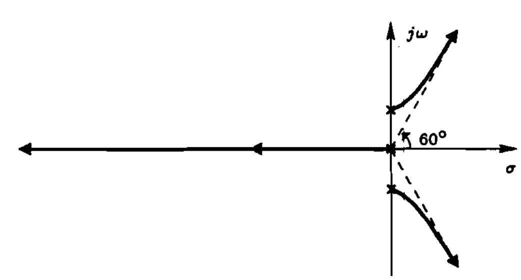
$$D(s) = \frac{1}{s^2+4^2}D'(s)$$

for some $D'(s)$. Since there is no other explicit requirement, we only need to ensure stability by a proper and simple choice of $D'(s)$.

Our choices increase in complexity as we include more zeros. Since the control already has two poles, we can include up to two zeros before we need to include another pole. As we can observe from the rough sketches of root-locus diagrams, the first two simplest choices, $D'(s) = K$ and $D'(s) = K(s+a)$ result in unstable systems.

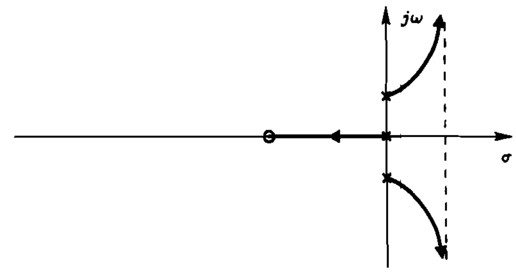
$D'(s) = K$ or

$$D(s)G(s) = K\frac{1}{s(s^2+4^2)} \text{ case.}$$



$D'(s) = K(s+a)$ or

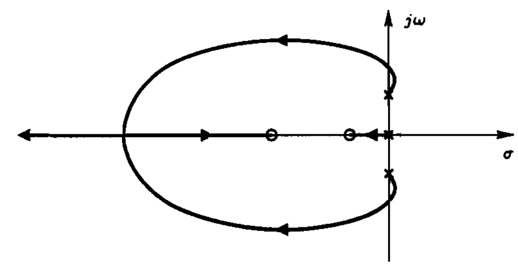
$D(s)G(s) = K\dfrac{s+a}{s(s^2+4^2)}$ case.



The next simplest choice $D'(s) = K(s+a)(s+b)$ has a chance to be asymptotically stable for some values of a, b, and K.

$D'(s) = K(s+a)(s+b)$ or

$D(s)G(s) = K\dfrac{(s+a)(s+b)}{s(s^2+4^2)}$ case.



To determine the requirements, we can use the Routh-Hurwitz's Table, where the characteristic equation is obtained from

$$1 + D(s)G(s) = 0,$$

or

$$1 + K\frac{(s+a)(s+b)}{s(s^2+16)} = 0,$$

$$s^3 + Ks^2 + \big(16 + K(a+b)\big)s + Kab = 0.$$

s^3	1	$16 + K(a+b)$
s^2	K	Kab
s	$\dfrac{K\big(16+K(a+b)\big) - Kab}{K}$	
1	Kab	

The Routh-Hurwitz's stability criterion implies the following conditions.

(a) $K > 0$.

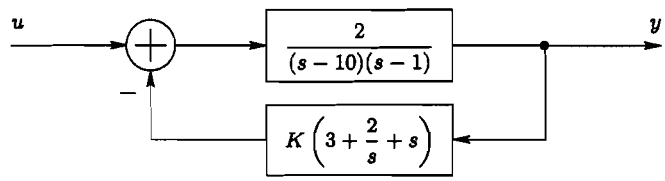
(b) $\left(K(16 + K(a+b)) - Kab\right)/K > 0$, or $K > (ab - 16)/(a+b)$ for $(a+b) > 0$.
(c) $Kab > 0$.

Therefore, one possible simplest controller is

$$D(s) = K\frac{(s+a)(s+b)}{s^2+16},$$

where a, b, $K > 0$, and $K > (ab - 16)/(a+b)$.

2. Consider the following control system.



Determine the range of the constant K, such that the system is asymptotically stable.

Solution: The stability of the closed-loop system can be determined using the Routh-Hurwitz's stability criterion on the characteristic polynomial. From the characteristic equation, $1 + G(s) = 0$, we have

$$1 + G(s)H(s) = 1 + \left(\frac{2}{(s-10)(s-1)}\right)\left(K\left(3 + \frac{2}{s} + s\right)\right)$$
$$= \frac{s^3 + (2K-11)s^2 + (6K+10)s + 4K}{s(s-10)(s-1)} = 0,$$

or

$$s^3 + (2K-11)s^2 + (6K+10)s + 4K = 0.$$

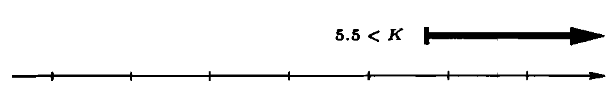
The Routh-Hurwitz's Table for the system becomes as given below.

s^3	1	$6K+10$
s^2	$2K-11$	$4K$
s	$\frac{(2K-11)(6K+10)-(4K)(1)}{2K-11}$	
1	$4K$	

The Routh-Hurwitz's stability criterion implies the following conditions.

(a) $2K - 11 > 0$.

$$K > 5.5.$$



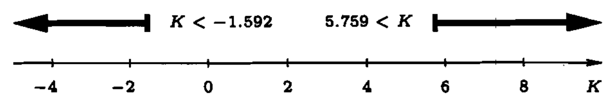
(b) $((2K-11)(6K+10)-(4K)(1))/(2K-11)>0.$

i. $2K-11>0$ Case:

$$12K^2-50K-110>0.$$
$$12(K-5.759)(K+1.592)>0,$$

or

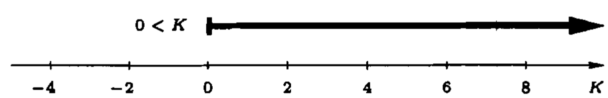
$$K<-1.592 \text{ or } K>5.759.$$



ii. $2K-11<0$ Case:
This case results in instability from the previous condition.

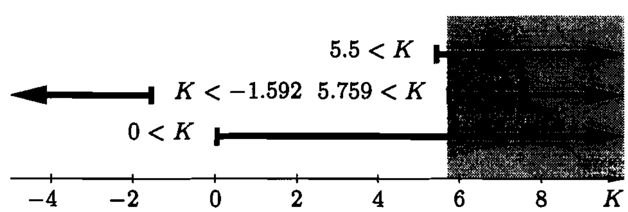
(c) $4K>0.$

$$K>0.$$



The intersection of all these regions leads to

$$K>5.759.$$



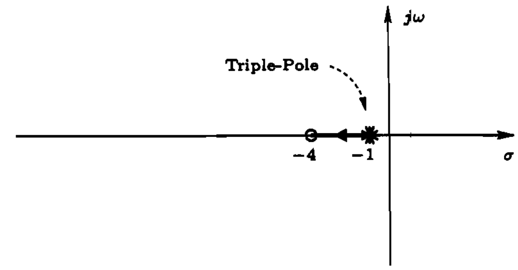
3. Consider the negative-feedback control-system with the following open-loop transfer-function. Construct the root-locus diagram. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angle of arrivals and/or departures.

$$G(s)=K\frac{s+4}{(s+1)^3}.$$

Solution: First, we sketch the pole-zero locations and the real-axis portion of the root-locus diagram. Then, we decide the important features to be determined.

Need to determine:

- Break-away angles,
- Asymptotes, and
- Imaginary-axis crossings.



Break-Away Angles: Equiangular Separation

At the break-away point, the poles separate from each other with equal angles. For a three-pole separation, the angle between any two adjacent poles is $360°/3 = 120°$. Since one of the poles branches towards the negative real-axis, the other poles will have $\pm 120°$ with respect to the negative real-axis or $\pm 60°$ with respect to the positive real-axis.

Asymptotes

Real-Axis Crossing: $\sigma_a = \dfrac{\sum p_i - \sum z_i}{n-m}$

The real-axis crossing of the asymptotes is at

$$\sigma_a = \frac{\sum_i p_i - \sum_i z_i}{n-m} = \frac{((-1)+(-1)+(-1)) - ((-4))}{3-1} = \frac{1}{2}.$$

Real-Axis Angles: $\theta_a = \pm(2k+1)\pi/(n-m)$

The angles that the asymptotes make with the real axis are determined from

$$\theta_a = \frac{\pm(2k+1)\pi}{n-m} = \frac{\pm(2k+1)\pi}{3-1} = \pm\frac{\pi}{2}.$$

Imaginary-Axis Crossings: Routh-Hurwitz's Table

The imaginary axis crossings can be determined from the Routh-Hurwitz's Table. From the characteristic equation,

$$1 + G(s) = 0,$$

$$1 + K\frac{s+4}{(s+1)^3} = 0,$$

$$s^3 + 3s^2 + (K+3)s + (4K+1) = 0,$$

we get the characteristic polynomial

$$q(s) = s^3 + 3s^2 + (K+3)s + (4K+1).$$

The Routh-Hurwitz's Table for this characteristic polynomial is given below.

s^3	1	$K+3$
s^2	3	$4K+1$
s	$\dfrac{(3)(K+3)-(1)(4K+1)}{3}$	
1	$4K+1$	

The imaginary-axis crossings will correspond to the positive values of K that would make a row of all zeros on the table. The first such candidate is the s-row. The s-row is all zero, when

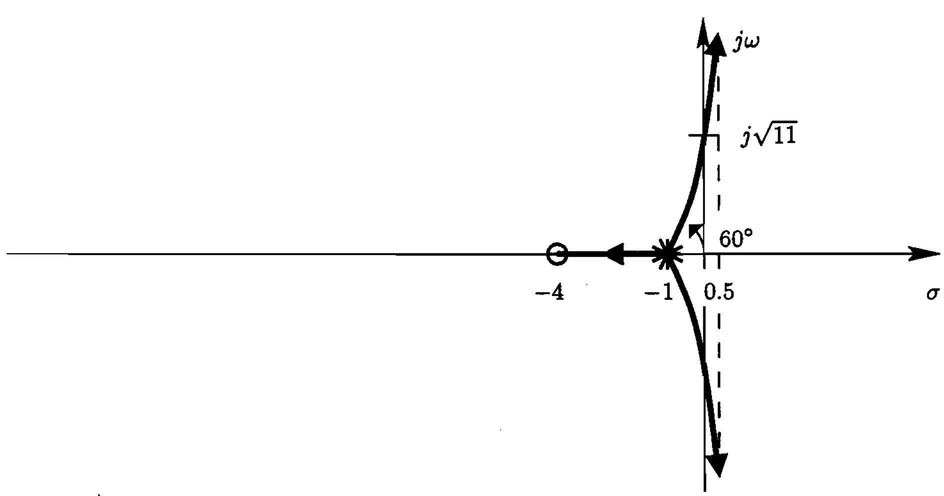
$$3(K+3)-(4K+1)=0,$$

or when $K=8$. For the positive value of K, we get a factor of the characteristic polynomial from the upper or the s^2-row. So,

$$\left((3)s^2+(4K+1)\right)_{K=8}=0,$$

or $s=\pm j\sqrt{11}$. Therefore, the imaginary-axis crossings are at $s\approx\pm j3.3166$.

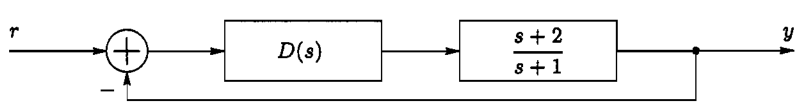
With the features determined, we can now sketch the root-locus diagram.



4. Sketch the location of the closed-loop poles for the following feedback control system under a proportional-integral controller

$$D(s)=1+\frac{K}{s}$$

for $K>0$.



Show all the important features.

Solution: The sketch of the location of the closed-loop poles is the root-locus diagram. However, in this case the open-loop gain of the system is

$$D(s)G(s) = \left(1 + \frac{K}{s}\right)\left(\frac{s+2}{s+1}\right) = \frac{(s+K)(s+2)}{s(s+1)},$$

where the root-locus variable K is not a multiplicative coefficient of the open-loop gain. So, we need to convert the problem into the conventional form while preserving the location of the closed-loop poles the same. The closed-loop poles are obtained from the characteristic equation, where

$$1 + D(s)G(s) = 0,$$

or

$$1 + \frac{(s+K)(s+2)}{s(s+1)} = 0,$$

$$\frac{s(s+1) + (s+K)(s+2)}{s(s+1)} = 0,$$

$$s(s+1) + (s+K)(s+2) = 0,$$

$$2s^2 + 3s + Ks + 2K = 0.$$

We need to regroup the characteristic equation, so that the characteristic equation is in the form

$$1 + K\frac{n(s)}{d(s)} = 0,$$

for some polynomials $n(s)$ and $d(s)$. So,

$$2s^2 + 3s + Ks + 2K = 0,$$

$$(2s^2 + 3s) + K(s+2) = 0,$$

$$\frac{(2s^2 + 3s) + K(s+2)}{(2s^2 + 3s)} = 0,$$

$$1 + K\frac{s+2}{2s^2 + 3s} = 0.$$

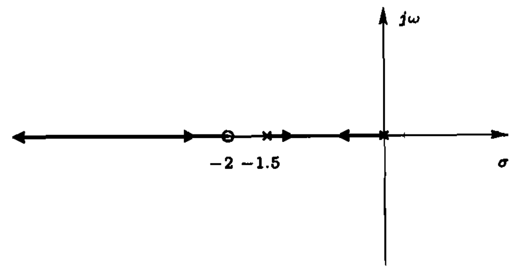
Therefore, the new open-loop gain

$$D'(s)G'(s) = K\frac{s+2}{2s^2 + 3s} = K\frac{s+2}{s(2s+3)}$$

generates the same closed-loop poles as the original open-loop gain, but the open-loop gain $D'(s)G'(s)$ of the new system is in the usual form for the generation of the root-locus diagram. In other words, the locations of the closed-loop poles based on the open-loop gains $D(s)G(s)$ and $D'(s)G'(s)$ are identical, however we can use the regular root-locus drawing techniques on the primed system.

First, we sketch the pole-zero locations and the real-axis portion of the root-locus diagram. Then, we decide the important features to be determined.

We observe that we have the two-pole one-zero case, where there will be a break-away and a break-in points on the real axis.



For the two-pole one-zero case, the portion of the root-locus diagram outside of the real axis is on a circle with the center at the zero,

$$\text{center} = z = -2,$$

and the radius that is the geometric mean of the distances of the poles from the zero,

$$\text{radius} = \sqrt{(p_1 - z)(p_2 - z)} = \sqrt{((-1.5) - (-2))((0) - (-2))} = 1.$$

Therefore,