Mathematics 6215 Homework Set 1 Name: Dr. Grow

Due Date: ____________

Solve the following problems which appeared on the Spring 2013 Math 315 Final Exam.

1. In each of the following, compute the Lebesgue integral of f over the set E or show that f is not integrable over E. The symbol $\mathbb{A}$ represents the set of algebraic numbers, $\mathbb{Q}$ stands for the set of rational numbers, and P denotes the Cantor ternary set. Please justify the steps in your computations.

(a) $f(x)=\begin{cases} -1 & \text{if } x\in P, \\ 3 & \text{if } x\in[0,1]\setminus P, \\ 2 & \text{if } x\in[-1,0]\setminus\mathbb{Q}, \\ -4 & \text{if } x\in[-1,0]\cap\mathbb{Q}. \end{cases}$ $\quad E=[-1,1]$

(b) $f(x)=\begin{cases} \cos(x) & \text{if } x\in\mathbb{A}, \\ \dfrac{1}{x} & \text{if } x\in[0,1]\setminus\mathbb{A}. \end{cases}$ $\quad E=[0,1]$

(c) $f(x)=\begin{cases} x^2 & \text{if } x\in\mathbb{Q}, \\ e^{-|x|}\sin(x) & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}. \end{cases}$ $\quad E=(-\infty,\infty)$

(d) $f(x)=\sum_{n=1}^{\infty}\dfrac{\sin(nx)}{n}$ $\quad E=[0,\pi]$

2. Let f be the function defined on the interval $[0,1]$ as follows: $f(x)=0$ if x is a point of the Cantor ternary set and $f(x)=1/k$ if x is in one of the complementary open intervals of the Cantor set with length 3^{-k}. For example, $f(1/3)=0, f(1/2)=1,$ and $f(4/5)=1/2$.

(a) Show that f is a Lebesgue measurable function.

(b) Evaluate $\int_{[0,1]} f(x)dx$.

In the following problems, let $\|f\|_{L^p}=\left(\int_{\mathbb{R}}|f(x)|^p dx\right)^{1/p}$ for $1\le p<\infty$ and $f\in L^p(\mathbb{R})$.

3. (a) Give an example of a sequence $\langle f_n\rangle$ of measurable functions on $\mathbf{R}$ with the following properties: $f_n(x)\to f(x)$ pointwise on $\mathbf{R}$, $\|f_n\|_{L^1}\le M<\infty$ for all $n\ge 1$, and $\|f_n-f\|_{L^1}$ does not converge to 0 as $n\to\infty$.

(b) If $\{f_n\}$ is a sequence of measurable functions which converges to f pointwise on $\mathbf{R}$ and $\|f_n\|_{L^1}\to M<\infty$, what can you conclude about $\|f\|_{L^1}$? Justify this conclusion with a proof.

(c) If $\{f_n\}$ is a sequence of measurable functions which converges to f pointwise on $\mathbf{R}$ and $\|f_n\|_{L^1}\to\|f\|_{L^1}<\infty$, show that $\|f_n-f\|_{L^1}\to 0$.

4. (a) Let f be Lebesgue integrable on $\mathbf{R}$. Show that

$$m(\{x \in \mathbf{R} : |f(x)| \geq \lambda\}) \leq \frac{\|f\|_{L^1}}{\lambda} \quad \text{for all } \lambda > 0.$$

(b) Let f be a measurable function on $\mathbf{R}$ with the property that there is a positive number C such that

$$m(\{x \in \mathbf{R} : |f(x)| \geq \lambda\}) \leq \frac{C}{\lambda} \quad \text{for all } \lambda > 0.$$

Is it true that $f \in L^1(\mathbf{R})$? Justify your answer.

(c) Generalize the results of (a) and (b) to $L^p(\mathbf{R})$ where $1 < p < \infty$.

#1. (a) $f = -1\chi_P + 3\chi_{[0,1]\setminus P} + 2\chi_{[-1,0]\setminus \mathbb{Q}} - 4\chi_{[-1,0]\cap \mathbb{Q}}$

Note that $m(P) = 0 = m([-1,0]\cap\mathbb{Q})$ so $m([0,1]\setminus P) = m[0,1] = 1$ and $m([-1,0]\setminus \mathbb{Q}) = m[-1,0] = 1$. Therefore

$$\int_{[-1,1]} f\,dm = \int_{[-1,1]} \left(-\chi_P + 3\chi_{[0,1]\setminus P} + 2\chi_{[-1,0]\setminus\mathbb{Q}} - 4\chi_{[-1,0]\cap\mathbb{Q}}\right)dm$$

$$= (-1)m(P) + 3m([0,1]\setminus P) + 2m([-1,0]\setminus\mathbb{Q}) - 4m([-1,0]\cap\mathbb{Q})$$

$$= 0 + 3\cdot 1 + 2\cdot 1 - 0$$

$$= \boxed{5}$$

(b) Since A is countable, $m(A)=0$ so $f(x) = \frac{1}{x}$ a.e. on $[0,1]$.

Therefore, if $f_n(x) = \min\{n, \frac{1}{x}\}$ $(n=1,2,3,\dots)$ then $f_n(x) \uparrow \frac{1}{x}$ on $[0,1]$ as $n\to\infty$, so

$$\int_{[0,1]} f\,dm = \int_{[0,1]} \frac{1}{x}\,dm \overset{\text{Monotone Convergence Theorem}}{=} \lim_{n\to\infty}\int_{[0,1]} f_n\,dm$$

$$= \lim_{n\to\infty}\left(\int_0^1 f_n\,dx\right)$$

Since f_n is continuous on $[0,1]$, the Lebesgue and Riemann integrals of f_n exist and agree.

$$= \lim_{n\to\infty}\left(\int_0^{1/n} f_n\,dx + \int_{1/n}^1 f_n\,dx\right)$$

$$= \lim_{n\to\infty}\left(\int_0^{1/n} n\,dx + \int_{1/n}^1 \frac{1}{x}\,dx\right)$$

$$= \lim_{n\to\infty}\left(n\cdot\frac{1}{n} + \ln(x)\Big|_{1/n}^{1}\right)$$

$$= 1 + \lim_{n\to\infty}\left[\overset{0}{\ln(1)} - \ln(1/n)\right]$$

Therefore $\int_{[0,1]} f\,dm = 1 + \lim_{n\to\infty} \ln(n) = \infty$. I.e. $f \notin L^1[0,1]$.

(c) Since $\mathbb{Q}$ is countable, $m(\mathbb{Q}) = 0$. Therefore $f(x) = e^{-|x|}\sin(x)$ a.e. on $\mathbb{R}$ and hence $\int_{\mathbb{R}} f\,dm = \int_{\mathbb{R}} e^{-|x|}\sin(x)\,dm$. Observe that $\left|e^{-|x|}\sin(x)\right| \le e^{-|x|}$ for all $x \in \mathbb{R}$. We claim that $x \overset{g}{\mapsto} e^{-|x|}$ belongs to $L^1(\mathbb{R})$. To see this, let $g_n(x) = e^{-|x|}\chi_{(-n,n)}(x)$ $(n=1,2,3,\dots)$. Then $0 \le g_n(x) \le g_{n+1}(x)$ for $n=1,2,3,\dots$ and all $x\in\mathbb{R}$ and $e^{-|x|} = \lim_{n\to\infty} g_n(x)$ for all $x \in \mathbb{R}$. By the Monotone Convergence Theorem

$$\int_{\mathbb{R}} e^{-|x|}\,dm = \lim_{n\to\infty}\int_{\mathbb{R}} e^{-|x|}\chi_{(-n,n)}(x)\,dm = \lim_{n\to\infty} 2\int_0^n e^{-x}\,dx = \lim\left(-2e^{-x}\Big|_{x=0}^{n}\right) = \lim_{n\to\infty}\left(2 - 2e^{-n}\right)$$
$$= 2 < \infty.$$

This proves the claim that $x \overset{g}{\mapsto} e^{-|x|}$ belongs to $L^1(\mathbb{R})$.

Note that $f_n(x) = e^{-|x|}\sin(x)\chi_{(-n,n)}(x)$ $(n=1,2,3,\dots)$ satisfies $e^{-|x|}\sin(x) = \lim_{n\to\infty} f_n(x)$ for $x\in\mathbb{R}$ and $|f_n(x)| \le e^{-|x|}$ for all $n \ge 1$ and $x \in \mathbb{R}$. By Lebesgue's Dominated Convergence Theorem, $x \mapsto e^{-|x|}\sin(x)$ belongs to $L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} e^{-|x|}\sin(x)\,dm = \lim_{n\to\infty}\int_{\mathbb{R}} e^{-|x|}\sin(x)\chi_{(-n,n)}(x)\,dm = \lim_{n\to\infty}\int_{-n}^{n} \underbrace{e^{-|x|}\sin(x)}_{\text{odd fcn.}}\,dx = \lim_{n\to\infty} 0 = 0.$$

Therefore $\int_{\mathbb{R}} f\,dm = \boxed{0}$.

(d) Consider the function $f(x) = \frac{\pi - x}{2}$ if $0 \le x < 2\pi$. Then

$$\hat{f}(0) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{4\pi}\int_0^{2\pi}(\pi - x)\,dx = -\frac{1}{4\pi}\frac{(\pi-x)^2}{2}\Bigg|_{x=0}^{2\pi} = -\frac{1}{8\pi}\left(\pi^2 - \pi^2\right) = 0.$$

If $n \neq 0$ then

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx = \frac{1}{4\pi}\int_0^{2\pi}\underbrace{(\pi - x)}_{v}\underbrace{e^{-inx}\,dx}_{dv} = \frac{1}{4\pi}\left\{(\pi - x)\frac{e^{-inx}}{-in}\Bigg|_0^{2\pi} - \cancel{\int_0^{2\pi}\frac{e^{-inx}}{-in}(-dx)}^{\,0}\right\}$$

$$= \frac{1}{-4\pi n i}\left\{-\pi - \pi\right\} = \frac{1}{2in}.$$

Therefore

$$\hat{f}(n)e^{inx} + \hat{f}(-n)e^{-inx} = \frac{1}{2in}e^{inx} + \frac{1}{2i(-n)}e^{-inx} = \frac{1}{n}\left(\frac{e^{inx} - e^{-inx}}{2i}\right) = \frac{1}{n}\sin(nx).$$

Hence $S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx} = \sum_{n=1}^{N} \frac{1}{n}\sin(nx)$. Clearly $|f(x) - f(y)| \le \frac{1}{2}|x-y|$ for all $0 < y < x < 2\pi$ so by Dini's Theorem (Rudin's P.M.A., Theorem 8.14) $S_N f(x) \longrightarrow f(x)$ on $0 < x < 2\pi$. That is,

$$\int_{[0,\pi]} \left\{\sum_{n=1}^{\infty} \frac{1}{n}\sin(nx)\right\} dm = \int_{[0,\pi]} \left(\frac{\pi - x}{2}\right) dm$$

$$= \frac{1}{2}\int_0^{\pi} (\pi - x)\,dx$$

$$= -\frac{1}{4}(\pi - x)^2\Big|_0^{\pi}$$

$$= \boxed{\frac{\pi^2}{4}}.$$

#2. (a) Let $I_{j,k}$ $(1 \le j \le 2^{k-1})$ denote the j^{th} interval, ordered from left to right, removed at the k^{th} stage of the construction of the Cantor set. For example,

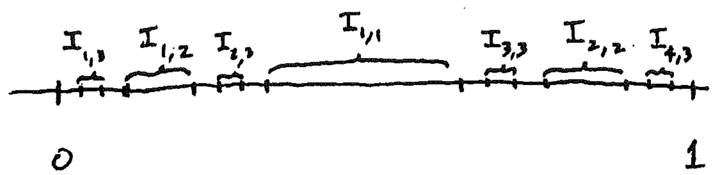
$$I_{1,1} = \left(\tfrac{1}{3}, \tfrac{2}{3}\right)$$

$$I_{1,2} = \left(\tfrac{1}{9}, \tfrac{2}{9}\right), \quad I_{2,2} = \left(\tfrac{7}{9}, \tfrac{8}{9}\right),$$

$$I_{1,3} = \left(\tfrac{1}{27}, \tfrac{2}{27}\right), \quad I_{2,3} = \left(\tfrac{7}{27}, \tfrac{8}{27}\right), \quad I_{3,3} = \left(\tfrac{19}{27}, \tfrac{20}{27}\right),$$

$$I_{4,3} = \left(\tfrac{25}{27}, \tfrac{26}{27}\right),$$

etc.



It is apparent from the definition of f that

$$f(x) = \sum_{k=1}^{\infty} \sum_{j=1}^{2^{k-1}} \frac{1}{k} \chi_{I_{j,k}}(x)$$

for all $0 \le x \le 1$. Therefore f, being the limit of a sequence of measurable simple functions, is measurable.

(b) $$\int_0^1 f(x)\,dx \overset{\text{M.C.T.}}{=} \sum_{k=1}^{\infty} \int_0^1 \sum_{j=1}^{2^{k-1}} \frac{1}{k} \chi_{I_{j,k}}(x)\,dx = \sum_{k=1}^{\infty} \sum_{j=1}^{2^{k-1}} \frac{1}{k} m(I_{j,k})$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{2^{k-1}} \frac{1}{k} \cdot 3^{-k} = \sum_{k=1}^{\infty} \frac{2^{k-1}}{k\,3^k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(2/3)^k}{k} = \boxed{\frac{\ln(3)}{2}}.$$

Note: $\sum_{k=0}^{\infty} t^k = \frac{1}{1-t}$ for $|t|<1$, and the series converges uniformly on each compact subset of $(-1,1)$. Consequently if $x \in (-1,1)$,

$$\sum_{j=1}^{\infty} \frac{x^j}{j} = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = \sum_{k=0}^{\infty} \int_0^x t^k\,dt = \int_0^x \Big(\sum_{k=0}^{\infty} t^k\Big) dt$$

$$= \int_0^x \frac{1}{1-t}\,dt = -\ln|1-t|\Big|_{t=0}^{x} = -\ln(1-x) = \ln\left(\frac{1}{1-x}\right).$$

From this, we see that $\sum_{j=1}^{\infty} \frac{(2/3)^j}{j} = \ln\left(\frac{1}{1-2/3}\right) = \ln(3)$.

#3. (a) Let $f_n = \chi_{[n,n+1)}$ for $n = 1, 2, 3, \ldots$ Then $f_n \to 0$ pointwise on $\mathbb{R}$, $\|f_n\|_{L^1} = 1$ for all $n \geq 1$, and $\|f_n - f\|_{L^1} = \|f_n - 0\|_{L^1}$ $= 1 \not\to 0$ as $n \to \infty$.

(b) Let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence of measurable functions which converges to f pointwise on $\mathbb{R}$ and $\|f_n\|_{L^1} \to M < \infty$. Then $\|f\|_{L^1} \leq M$.

Proof: By Fatou's Lemma,

$$\int_{\mathbb{R}} |f|\,dx = \int_{\mathbb{R}} \liminf_{n\to\infty} |f_n(x)|\,dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}} |f_n(x)|\,dx = M. \quad \text{Q.E.D.}$$

(Note: The inequality in the conclusion of (b) may be strict as part (a) shows.)

(c) Let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence of measurable functions which converges to f pointwise on $\mathbb{R}$ and $\|f_n\|_{L^1} \to \|f\|_{L^1} < \infty$. Consider

$$g_n = |f_n| + |f| - |f_n - f| \qquad (n = 1, 2, 3, \ldots).$$

Since $|f_n - f| \leq |f_n| + |f|$, it follows that $g_n \geq 0$ for all $n \geq 1$. Clearly each g_n is measurable and $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \left(|f_n(x)| + |f(x)| - |f_n(x) - f(x)| \right)$ $= 2|f(x)|$ for $x \in \mathbb{R}$. By Fatou's Lemma,

$$\int_{\mathbb{R}} 2|f|\,dx = \int_{\mathbb{R}} \left(\liminf_{n\to\infty} g_n(x) \right) dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}} g_n(x)\,dx$$

$$= \liminf_{n \to \infty} \int_{\mathbb{R}} \Big(|f_n(x)| + |f(x)| - |f_n(x) - f(x)| \Big) dx$$

$$= \int_{\mathbb{R}} 2|f|dx + \liminf_{n \to \infty} \Big(- \int_{\mathbb{R}} |f_n - f| \, dx \Big)$$

$$= \int_{\mathbb{R}} 2|f|dx - \limsup_{n \to \infty} \int_{\mathbb{R}} |f_n - f| dx .$$

But $f \in L^1(\mathbb{R})$, so cancelling and rearranging gives

$$\limsup_{n \to \infty} \| f_n - f \|_{L^1} \leq 0 .$$

On the other hand $0 \leq \liminf_{n \to \infty} \| f_n - f \|_{L^1}$ so $\lim_{n \to \infty} \| f_n - f \|_{L^1}$ exists and is equal to zero.

#4. (a) Fix $\lambda > 0$ and let $E_\lambda = \{ x \in \mathbb{R} : |f(x)| \geq \lambda \}$. Then

$$\lambda m(E_\lambda) = \int_{\mathbb{R}} \lambda \chi_{E_\lambda} \, dm \leq \int_{\mathbb{R}} |f| \chi_{E_\lambda} \, dm \leq \int_{\mathbb{R}} |f| \, dm = \| f \|_{L^1} .$$

Therefore $m(\{x \in \mathbb{R} : |f(x)| \geq \lambda\}) \leq \dfrac{\|f\|_{L^1}}{\lambda}$ for all $\lambda > 0$.

(b) No, f need not be in $L^1(\mathbb{R})$ for consider $f(x) = \frac{1}{x} \chi_{(0,\infty)}(x)$ on $\mathbb{R}$. Clearly $f \notin L^1(\mathbb{R})$. (This follows from the Monotone Convergence Theorem and the fact that $\int_{[\frac{1}{n}, n]} \frac{1}{x} \, dm = 2 \ln(n)$ for $n \geq 1$.) However, if $x > 0$ and $\lambda > 0$, then $\frac{1}{x} \geq \lambda$ if and only if $\frac{1}{\lambda} \geq x > 0$. Therefore

$$m(\{x \in \mathbb{R} : |f(x)| \geq \lambda\}) = m((0, \tfrac{1}{\lambda}]) = \frac{1}{\lambda}$$

for all $\lambda > 0$.

(c) Let $f \in L^p(\mathbb{R})$ for $1 < p < \infty$ and let $\lambda > 0$. Define $E_\lambda = \{ x \in \mathbb{R} : |f(x)| \geq \lambda \}$. Then

$$\lambda^p m(E_\lambda) = \int_{\mathbb{R}} \lambda^p \chi_{E_\lambda} \, dm \leq \int_{\mathbb{R}} |f|^p \chi_{E_\lambda} \, dm \leq \int_{\mathbb{R}} |f|^p dm = \|f\|_{L^p}^p .$$

Thus $m(E_\lambda) \leq \dfrac{\|f\|_{L^p}^p}{\lambda^p}$ for all $\lambda > 0$.

On the other hand, consider $f(x) = \dfrac{1}{|x|^{1/p}}$ for $x \in \mathbb{R}$. Then clearly $f \notin L^p(\mathbb{R})$, but for all $\lambda > 0$

$$\lambda^p m(E_\lambda) = \lambda^p m\left(\left[-\frac{1}{\lambda^p}, \frac{1}{\lambda^p}\right]\right) = \lambda^p \cdot \frac{2}{\lambda^p} = 2 .$$

Thus, $m(E_\lambda) \leq \dfrac{2}{\lambda^p}$ for all $\lambda > 0$.