Mathematics 6215 Homework Set 2 Name: Dr. Grow

Solve the following problems from Royden and Fitzpatrick's *Real Analysis*.

p. 90 # 36

p. 95 # 40, 44, 47

Also, solve each of these problems:

A. Let E $\subseteq \mathbf{R}$ and $f:\mathbf{R}\to[-\infty,\infty]$. Show that:

(1) there exists a sequence $\{s_n\}$ of simple functions, measurable if f is, such the $s_n(x)\to f(x)$ for all x in E;

(2) if f is bounded, then the sequence $\{s_n\}$ in (1) can be constructed so that the convergence to f is uniform on E;

(3) if $f\geq 0$, then the sequence $\{s_n\}$ in (1) can be constructed nonnegative and pointwise increasing: $0\leq s_1(x)\leq s_2(x)\leq \ldots$ for all x in E.

B. Let f be a bounded nonnegative function on $[a,b]$ which is Riemann integrable on that interval.

(1) Show that there exist sequences $\{\varphi_n\}$ and $\{\theta_n\}$ of step functions on $[a,b]$ such that $0\leq\theta_n\leq f\leq\varphi_n$ and

$$\int_a^b [\varphi_n(x)-\theta_n(x)]\,dx \leq \frac{1}{n}.$$

(2) Let $\theta(x)=\sup_n \theta_n(x)$ and $\varphi(x)=\sup_n \varphi_n(x)$ for x in $[a,b]$. Show that θ and φ are are measurable, $\theta\leq f\leq\varphi$, and $m(\{x\in[a,b]:\theta(x)<\varphi(x)\})=0$.

(3) Show that $\theta=\varphi$ a.e. on $[a,b]$ and that f is a measurable function.

(4) Show that the Riemann integral of f on $[a,b]$ is equal to the Lebesgue integral of f on $[a,b]$.

Notation for problem C: Let $f\in L^1(-\infty,\infty)$. The Fourier transform of f at any real number y is given by

$$\hat{f}(y)=\int_{(-\infty,\infty)} f(x)e^{-ixy}\,dx.$$

C. Let $f\in L^1(-\infty,\infty)$. Show that:

(1) $\hat{f}$ is a continuous function on $\mathbf{R}$;

(2) if $g(x)=xf(x)$ is in $L^1(-\infty,\infty)$, then $\hat{f}$ is differentiable on $\mathbf{R}$ with

$$\hat{f}'(y)=-i\int_{(-\infty,\infty)} xf(x)e^{-ixy}\,dx \qquad (-\infty<y<\infty).$$

C. (1) Let $f \in L^1(\mathbb{R})$ and recall

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dm(x) \qquad (\xi \in \mathbb{R}).$$

Fix $\xi \in \mathbb{R}$ and let $\langle \xi_n \rangle_{n=1}^{\infty}$ be any sequence of real numbers converging to ξ. Define measurable functions $f_n : \mathbb{R} \to \mathbb{C}$ $(n=1,2,3,\dots)$ by

$$f_n(x) = f(x) e^{-ix\xi_n} \qquad (x \in \mathbb{R}).$$

Observe that

(1) $\lim_{n\to\infty} f_n(x) = f(x)e^{-ix\xi}$ for all $x \in \mathbb{R}$, and

(2) $|f_n(x)| = |f(x)|$ for all $x \in \mathbb{R}$ and $n \geq 1$.

Since $|f| \in L^1(\mathbb{R})$, we may apply Lebesgue's Dominated Convergence Theorem to obtain

$$\lim_{n\to\infty} \hat{f}(\xi_n) = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x) dm(x) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dm(x) = \hat{f}(\xi).$$

That is, $\hat{f}$ is continuous on $\mathbb{R}$.

(2) Let $f \in L^1(\mathbb{R})$ and the function $x \overset{g}{\longmapsto} x f(x)$ be in $L^1(\mathbb{R})$ as well. Fix $y \in \mathbb{R}$ and let $\langle h_n \rangle_{n=1}^{\infty}$ be any sequence of nonzero real numbers converging to zero. Consider the limit

$$\hat{f}'(y) = \lim_{n\to\infty} \frac{\hat{f}(y+h_n) - \hat{f}(y)}{h_n} = \lim_{n\to\infty} \int_{\mathbb{R}\setminus\{0\}} x f(x) e^{-iyx} \left[\frac{e^{-ih_n x} - 1}{h_n x} \right] dm(x).$$

Observe that if $\theta \neq 0$ then

$$\left| \frac{e^{-i\theta} - 1}{\theta} \right| = \sqrt{\frac{(1-\cos\theta)^2 + \sin^2\theta}{\theta^2}} = \sqrt{\frac{2 - 2\cos(\theta)}{\theta^2}} = \sqrt{\frac{4\sin^2(\theta/2)}{\theta^2}} = \left| \frac{\sin(\theta/2)}{\theta/2} \right| \leq 1.$$

Furthermore,

$$\lim_{\theta\to 0} \frac{e^{-i\theta}-1}{\theta} = \lim_{\theta\to 0} \frac{-ie^{-i\theta}}{1} = -i.$$

It follows that, for a fixed $y \in \mathbb{R}$,

$$g_n(x) = x f(x) e^{-iyx}\left[\frac{e^{-ih_n x}-1}{h_n x}\right] \qquad (x \in \mathbb{R}\setminus\{0\},\ n=1,2,3,\ldots)$$

defines a sequence of measurable functions such that

$$(1) \quad \lim_{n\to\infty} g_n(x) = -ixf(x)e^{-iyx} \qquad (x \in \mathbb{R}\setminus\{0\})$$

$$\text{and } (2) \quad |g_n(x)| = |xf(x)|\cdot\left|\frac{e^{-ih_n x}-1}{h_n x}\right| \le |xf(x)| \qquad (x\in\mathbb{R}\setminus\{0\}, n\ge 1).$$

Because $g(x) = xf(x)$ belongs to $L^1(\mathbb{R})$, we may apply Lebesgue's Dominated Convergence Theorem to obtain

$$\hat{f}'(y) = \lim_{n\to\infty} \int_{\mathbb{R}\setminus\{0\}} g_n(x)\,dm(x) = -i\int_{\mathbb{R}} xf(x)e^{-iyx}\,dm(x) \qquad (y\in\mathbb{R}).$$

In particular, $\hat{f}$ is differentiable on $\mathbb{R}$.