Mathematics 6215 Homework Set 4 Name: Dr. Grow

Due Date: ____________

1. Let f and g be real functions in $L^2(0,1)$ with $\int_0^1 f(x)dx = 0$. Show that

$$\left(\int_0^1 f(x)g(x)dx\right)^2 \le \left(\int_0^1 f^2(x)dx\right)\left(\int_0^1 g^2(x)dx - \left(\int_0^1 g(x)dx\right)^2\right).$$

2. Let $E \subseteq (-\infty,\infty)$ be measurable with $m(E) < \infty$ and let $\{c_n\}_{n=1}^{\infty}$ be any real sequence. Show that

$$\lim_{n\to\infty} \int_E \cos^2(nx + c_n)dx = \frac{m(E)}{2}.$$

(Hints: You might find the Riemann-Lebesgue lemma and the identity $2\cos^2(A) = 1 + \cos(2A)$ useful.)

3. Let $\{f_n\}_{n=1}^{\infty}$ be a convergent sequence of functions in $L^p(0,1)$ for some $p \in [1,\infty)$, say $\|f - f_n\|_p \to 0$ as $n \to \infty$.

(a) Show that $\{f_n\}$ need not converge pointwise to f a.e. on $(0,1)$.

(b) Show that there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges pointwise to f a.e. on $(0,1)$.

(Hints: (a) Write $n = 2^m + k$ where $0 \le k < 2^m$, and let f_n be the characteristic function of the interval $(k2^{-m}, (k+1)2^{-m})$. (b) Examine the proof of the Riesz-Fischer theorem.)

4. Determine which of the following sequences defined on the integers are the Fourier coefficients of a function in $L^2(0,1)$. Justify your answers.

(a) $a_n = \dfrac{(-1)^n}{1+|n|}$

(b) $b_n = \dfrac{1}{\sqrt{(|n|+2)\ln(|n|+2)}}$

(c) $c_n = \sin(2n)\left(\sqrt[n^2+1]{2} - 1\right)^{|n|/2}$

(d) $d_n = \dfrac{1}{\sqrt{|n|}}$ if $|n| = 2^m$ for some integer $m \ge 0$; $d_n = 0$ otherwise.

Notation and Definitions: Let H denote the vector space of 1-periodic functions f on **R** with the property that the restriction of f to $(0,1)$ belongs to $L^2(0,1)$. For $f \in H$, define

$$\|f\| = \left(\int_0^1 |f(x)|^2\, dx\right)^{1/2}$$

and

$$\hat{f}(n) = \langle f, e_n \rangle = \int_0^1 f(x)e^{-2\pi i n x}dx \quad (n = 0, \pm 1, \pm 2, \ldots).$$

If f and g belong to H, define their convolution by

$$(f * g)(x) = \int_0^1 f(y)g(x-y)dy \quad (x \in \mathbf{R}).$$

5. (a) Show that the restriction mapping $f \mapsto f|_{(0,1)}$ is an isometry from $(H, \|\circ\|)$ onto $(L^2(0,1), \|\circ\|_2)$.

   (b) Show that if f and g belong to H, then so does $f * g$. (Don't forget to check measurability of $f * g$ on $(-\infty, \infty)$.)

   (c) If $f \in H$ and $n \in \mathbf{Z}$, show that $(f * e_n)(x) = \hat{f}(n) e_n(x)$ for all $x \in (-\infty, \infty)$.

6. Let $f \in H$.

   (a) Find the Fourier coefficients of the translate f_t of f.

   (b) Express the norm, $\|f_t - f\|$, in terms of the Fourier coefficients of f and the number t.

   (c) Show that $\liminf_{t \to 0^+} \frac{\|f_t - f\|}{t} > 0$ unless f is constant almost everywhere.

#1. Let $a = \int_0^1 g(x)dx$. Then $a\int_0^1 f(x)dx = 0$ so

$$\left(\int_0^1 f(x)g(x)dx\right)^2 = \left(\int_0^1 f(x)g(x)dx - a\int_0^1 f(x)dx\right)^2$$

$$= \left(\int_0^1 f(x)(g(x)-a)dx\right)^2$$

$$\overset{\text{C-B-S}}{\leq} \left(\int_0^1 f^2(x)dx\right)\cdot\left(\int_0^1 (g(x)-a)^2dx\right)$$

$$= \left(\int_0^1 f^2(x)dx\right)\cdot\left(\int_0^1 g^2(x)dx - 2a\underbrace{\int_0^1 g(x)dx}_{a} + a^2\right)$$

$$= \left(\int_0^1 f^2(x)dx\right)\cdot\left(\int_0^1 g^2(x)dx - \Big(\underbrace{\int_0^1 g(x)dx}_{a}\Big)^2\right)$$

#2. $$\lim_{n\to\infty}\int_E \cos^2(nx+c_n)dx = \lim_{n\to\infty}\int_E\left[\frac{1}{2}+\frac{1}{2}\cos(2nx+2c_n)\right]dx$$

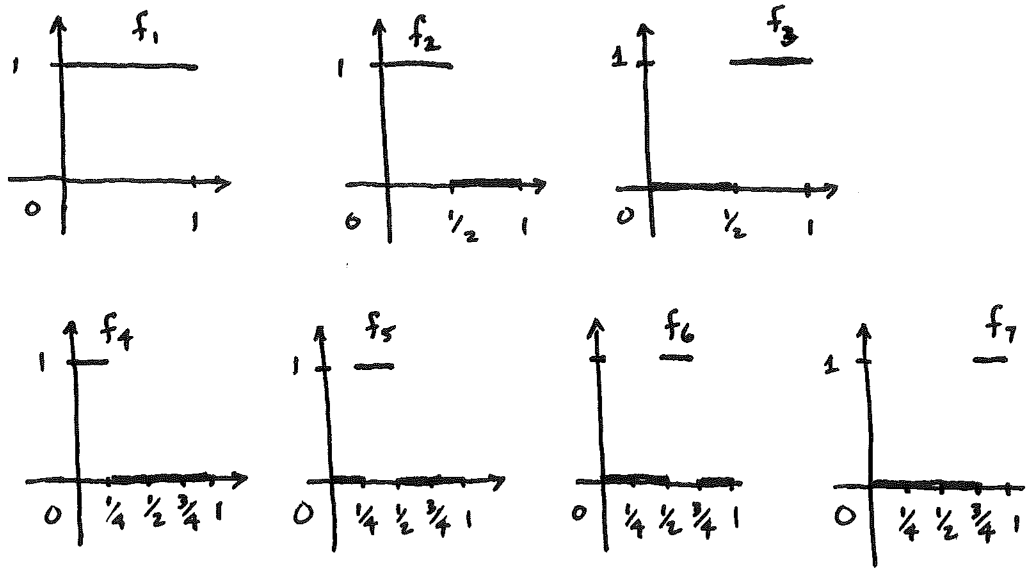
$$= \frac{m(E)}{2} + \lim_{n\to\infty}\int_{-\infty}^{\infty}\chi_E(x)\left[\frac{1}{2}\cos(2c_n)\cos(2nx) - \frac{1}{2}\sin(2c_n)\sin(2nx)\right]dx$$

$$= \frac{m(E)}{2} + \lim_{n\to\infty}\Big(\underbrace{\tfrac{1}{2}\cos(2c_n)}_{\text{bounded}}\underbrace{\int_{-\infty}^{\infty}\chi_E(x)\cos(2nx)dx}_{\text{goes to zero by Riemann-Lebesgue}}\Big) - \lim_{n\to\infty}\Big(\underbrace{\tfrac{1}{2}\sin(2c_n)}_{\text{bounded}}\underbrace{\int_{-\infty}^{\infty}\chi_E(x)\sin(2nx)dx}_{\text{goes to zero by Riemann-Lebesgue}}\Big)$$

$$= \frac{m(E)}{2}$$

#3(a) Write $n = 2^m + k$ where m and k are integers with $m \geq 0$ and $0 \leq k < 2^m$.

Let $f_n(x) = \chi_{\left(\frac{k}{2^m}, \frac{k+1}{2^m}\right)}(x)$ for x in $(0,1)$.



etc.

$\langle f_n(x) \rangle_{n=1}^{\infty}$ does not converge for any x in $(0,1)$ which is not a dyadic rational (i.e. for any x which is not of the form $\frac{p}{2^m}$ for some integers $p \geq 0$ and $m \geq 1$) because for such x-values there are infinitely many indices n for which $f_n(x) = 0$ and infinitely many indices n for which $f_n(x) = 1$. However, $\| f_n \|_p^p = 2^{-m}$ where $n = 2^m + k$, so $\lim_{n \to \infty} \| f_n \|_p = 0$ for all $p \in [1, \infty)$. That is, $f_n \to 0$ in $L^p(0,1)$.

(b) Since $\langle f_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence in $L^p(0,1)$, there is a subsequence $\langle f_{n_k} \rangle_{k=1}^{\infty}$ such that $\| f_{n_{k+1}} - f_{n_k} \|_p < \frac{1}{2^k}$, $k = 1, 2, 3, \ldots$ The proof of the Riesz-Fischer Theorem then shows that

$\lim_{k\to\infty} f_{n_k}(x)$ exists a.e. in $(0,1)$, say $g(x) = \lim_{k\to\infty} f_{n_k}(x)$ where this limit exists in $(0,1)$ and $g(x)=0$ otherwise. Then $g \in L^p(0,1)$ (again by the proof of the Riesz-Fischer Theorem) and $\| g-f \|_p \le \| g - f_{n_k} \|_p + \| f_{n_k} - f \|_p \to 0$ as $k\to\infty$ so $\|g-f\|_p = 0$. Thus $g(x) = f(x)$ a.e. in $(0,1)$; i.e. $f(x) = \lim_{k\to\infty} f_{n_k}(x)$ a.e. in $(0,1)$.

#4. (a) $\displaystyle\sum_{n=-\infty}^{\infty} |a_n|^2 = \sum_{n=-\infty}^{\infty} \frac{1}{(1+|n|)^2} \le 1 + 2\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ (p-series with $p=2>1$.)

Therefore $\langle a_n \rangle_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$ so there exists $f \in L^2(0,1)$ such that $\hat{f}(n) = a_n$ for all $n \in \mathbb{Z}$.

(b) $\displaystyle\sum_{n=-\infty}^{\infty} |b_n|^2 = \sum_{n=-\infty}^{\infty} \frac{1}{(|n|+2)\ln(|n|+2)} = \frac{1}{2\ln(2)} + \underbrace{2\sum_{k=3}^{\infty} \frac{1}{k\ln(k)}}_{\text{Divergent by the integral test.}}$.

$$\left(\int_3^\infty \frac{1}{x\ln(x)}\,dx = \lim_{M\to\infty} \int_{\ln(3)}^{\ln(M)} \frac{1}{u}\,du = \lim_{M\to\infty} \left[\ln(\ln(M)) - \ln(\ln(3)) \right] = \infty. \right)$$

Therefore $\langle b_n \rangle_{n=-\infty}^{\infty} \notin \ell^2(\mathbb{Z})$ so there is no function $f \in L^2(0,1)$ such that $\hat{f}(n) = b_n$ for all $n \in \mathbb{Z}$.

(c) $\displaystyle\sum_{n=-\infty}^{\infty} |c_n|^2 = \sum_{n=-\infty}^{\infty} \sin^2(2n)\left(\sqrt[n^2+1]{2} - 1\right)^{|n|} \le \underbrace{2\sum_{n=1}^{\infty} \left(\sqrt[n^2+1]{2} - 1\right)^n}_{\text{Convergent by the root test.}}$

$$\left[\sqrt[n]{\left(\sqrt[n^2+1]{2} - 1\right)^n} = \sqrt[n^2+1]{2} - 1 \to 0 \text{ as } n\to\infty. \right]$$

Since $\langle c_n \rangle_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$, there is a function $f \in L^2(0,1)$ such that $\hat{f}(n) = c_n$ for all $n \in \mathbb{Z}$.

(d) $d_{\pm 2^m} = \frac{1}{2^{m/2}}$ if $m = 0, 1, 2, 3, \ldots$ and $d_n = 0$ otherwise.

Therefore $\sum_{n=-\infty}^{\infty} |d_n|^2 = 2 \sum_{m=0}^{\infty} \frac{1}{2^m} = 4 < \infty$. Since $\langle d_n \rangle_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$, there is a function $f \in L^2(0,1)$ (in fact, $f \in C[0,1]$) such that $\hat{f}(n) = d_n$ for all $n \in \mathbb{Z}$.

#5 (a) clear.

(b) Let $f, g \in H$ and $x \in \mathbb{R}$. Then

$$(f * g)(x+1) = \int_0^1 f(y) g(x+1-y)\,dy \overset{\text{1-periodicity of } g}{=} \int_0^1 f(y) g(x-y)\,dy = (f*g)(x)$$

so $f*g$ is 1-periodic on $\mathbb{R}$. If $t \in \mathbb{R}$ then $f_t \in H$, $\lim_{t \to 0} \|f_t - f\| = 0$, and $f*g$ is (uniformly) continuous on $\mathbb{R}$. (The proof of the last two assertions is basically the same as in problem C of HW Set # 3.) Consequently $f*g$ is measurable on $\mathbb{R}$ and the restriction of $f*g$ to $(0,1)$ belongs to $L^2(0,1)$ so $f*g \in H$.

(c) Let $f \in H$ and $n \in \mathbb{Z}$. Then for all $x \in \mathbb{R}$,

$$(f * e_n)(x) = \int_0^1 f(y) e_n(x-y)\,dy = \int_0^1 f(y) e^{2\pi n i (x-y)}\,dy$$

$$= e^{2\pi n i x} \int_0^1 f(y) e^{-2\pi i n y}\,dy = \hat{f}(n) e_n(x).$$

#6. (a) Let $f \in H$ and $n \in \mathbb{Z}$. Then for all $t \in \mathbb{R}$,

$$\hat{f}_t(n) = \int_0^1 f_t(x) e^{-2\pi i n x}\,dx = \int_0^1 f(x-t) e^{-2\pi i n x}\,dx = \int_{-t}^{1-t} f(y) e^{-2\pi i n (y+t)}\,dy$$

$$= e^{-2\pi i n t} \int_{-t}^{1-t} f(y) e^{-2\pi i n y}\,dy .$$

But $\int_{-t}^{0} f(y) e^{-2\pi i n y}\,dy = \int_{1-t}^{1} f(y) e^{-2\pi i n y}\,dy$ by 1-periodicity of the integrand, so

$$\hat{f}_t(n) = e^{-2\pi i n t} \int_0^1 f(y) e^{-2\pi i n y}\,dy = \hat{f}(n) e^{-2\pi i n t} .$$

(b)
$$\| f_t - f \| = \left(\int_0^1 |f_t(x) - f(x)|^2\,dx \right)^{1/2}$$

$$= \left(\sum_{n=-\infty}^{\infty} |\widehat{(f_t - f)}(n)|^2 \right)^{1/2}$$

$$= \left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 |1 - e^{-2\pi i n t}|^2 \right)^{1/2}$$

$$= \left(\sum_{n=-\infty}^{\infty} 4 |\hat{f}(n)|^2 \sin^2(\pi n t) \right)^{1/2}$$

$$|1 - e^{-2i\theta}|^2$$
$$= (1 - \cos(2\theta))^2 + \sin^2(2\theta)$$
$$= 2 - 2\cos(2\theta)$$
$$= 2(1 - \cos(2\theta))$$
$$= 4 \sin^2(\theta)$$

(c) Suppose f is not constant almost everywhere. Then there exists a nonzero $n_0 \in \mathbb{Z}$ such that $\hat{f}(n_0) \neq 0$. (Otherwise $\hat{f}(n) = 0$ for all $0 \neq n \in \mathbb{Z}$ implies $0 = \lim_{N\to\infty} \| f - S_N f \| = \| f - \hat{f}(0) \|$ so $f = \hat{f}(0)$ a.e.)

Consequently,

$$\liminf_{t\to 0^+} \frac{\| f_t - f \|}{t} = \liminf_{t\to 0^+} \frac{\left(\sum_{n=-\infty}^{\infty} 4 |\hat{f}(n)|^2 \sin^2(n\pi t) \right)^{1/2}}{t} \geq \liminf_{t\to 0^+} \frac{2 |\hat{f}(n_0)| |\sin(n_0 \pi t)|}{t}$$

$$= 2 |\hat{f}(n_0)| |n_0| \pi > 0 .$$