Mathematics 415 Midterm Exam Name:______________________________

Fall 2013

1.(32 pts.) [3] (a) Define what it means for a subset E of $\mathbb{R}$ to be Lebesgue measurable.

[3] (b) Define the phrase "$\mathcal{A}$ is a $\sigma-$algebra of subsets of X".

[5] (c) Give three examples of $\sigma-$algebras of subsets of $\mathbb{R}$.

[3] (d) Define what it means for a function $f : E \to [-\infty, \infty]$ to be measurable.

[3] (e) Define the Lebesgue integral of a measurable function $f : E \to [0, \infty]$.

[5] (f) State the Monotone Convergence Theorem.

[5] (g) State Fatou's Lemma.

[5] (h) State the Dominated Convergence Theorem.

2.(26 pts.) Let p be a positive real number and let E be a measurable subset of $\mathbb{R}$.

[3] (a) Define the space $L^p(E)$.

[3] (b) Define the space $L^\infty(E)$.

[5] (c) State Holder's inequality and conditions under which Holder's inequality is valid.

[5] (d) State Minkowski's inequality and conditions under which Minkowski's inequality holds.

[5] (e) Give an example of a function in $L^2(0,1)$ which is not in $L^1(0,1)$, or tell why this is not possible.

[5] (f) Give an example of a function in $L^2(-\infty,\infty)$ which is not in $L^1(-\infty,\infty)$, or tell why this is not possible.

3.(28 pts.) Let $(X, \|\cdot\|_X)$ be a real normed vector space and let $(Z, \langle\cdot,\cdot\rangle_Z)$ be a real inner product space.

[5] (a) Give an example of an inner product space that is not a normed vector space, or tell why this is not possible.

[3] (b) Define the phrase "$\langle x_n \rangle_{n=1}^\infty$ is a Cauchy sequence in X."

[3] (c) Define the phrase "$\langle x_n \rangle_{n=1}^\infty$ is a convergent sequence in X."

[5] (d) Give an example of a Cauchy sequence, in an appropriate normed vector space, which is not a convergent sequence, or tell why this is not possible.

[3] (e) Define what it means for the space $(X, \|\cdot\|_X)$ to be complete.

[3] (f) What is a Banach space?

[3] (g) What is a Hilbert space?

[3] (h) Define the phrase "Λ is a bounded linear functional on X."

4.(25 pts.) [10] (a) State the Riesz-Fischer Theorem.

[5] (b) State a theorem characterizing the bounded linear functionals on $\left(C[0,1], \|\cdot\|_u\right)$

[10] (c) State a theorem characterizing the bounded linear functionals on $L^p(E)$.

5.(39 pts.) [3] (a) Define the Fourier transform of a function f in $L^1(0,1)$.

[3] (b) Define the Nth partial sum of the Fourier series of a function f in $L^1(0,1)$.

[10] (c) State a theorem about convergence of the Fourier series of functions in an appropriate L^p space.

[3] (d) Define the space $l^p(\mathbf{Z})$ where $0 < p < \infty$.

[5] (e) State Bessel's Inequality and conditions under which it holds.

[5] (f) State Parseval's Identity and conditions under which it holds.

[10] (g) State a theorem giving an isometry between an appropriate L^p space and an appropriate l^q space.