Mathematics 322 — Final Exam, Spring 2005 — Name:________________ (2 pts.)

You may use your prepared sheet of formulas on this examination, but no books, notes, or other aids. Please turn in your prepared sheet of formulas with your name printed at the top; you will receive 2 points on this exam for doing so. You will have 120 minutes to complete your solutions.

1.(16 pts.) If $\mathbf{v} \approx (v_x, v_y, v_z)$ belongs to E^3 define $\mathbf{Tv} \approx (-2v_x + 3v_z, -v_z, v_x + 2v_y)$.

(a) Why is $\mathbf{T}$ a tensor?

(b) Determine the matrix of Cartesian components of the tensor $\mathbf{T}$.

(c) Complete the blanks.

$$\mathbf{Sv} \equiv \tfrac{1}{2}(\mathbf{T} + \mathbf{T}^{\mathrm{T}})\mathbf{v} \approx$$

$$\mathbf{Av} \equiv \tfrac{1}{2}(\mathbf{T} - \mathbf{T}^{\mathrm{T}})\mathbf{v} \approx$$

(d) Determine the matrices of the Cartesian components of $\mathbf{S}$ and $\mathbf{A}$.

(e) Find the Cartesian components of the vector $\mathbf{w}$ such that $\mathbf{Av} = \mathbf{w} \times \mathbf{v}$ for all $\mathbf{v}$ in E^3.

2.(16 pts.) Consider the transformation of coordinates

$$x = u - v^2$$

$$y = u + v$$

in E^2 where $-\infty < u < \infty$ and $-\tfrac{1}{2} < v$. Compute:

(a) the base vectors $\mathbf{g}_1 = \mathbf{g}_u$ and $\mathbf{g}_2 = \mathbf{g}_v$;

(b) the reciprocal base vectors;

(c) the six Christoffel symbols $\Gamma^1_{11} = \Gamma^u_{uu}$, $\Gamma^1_{12} = \Gamma^u_{uv}$, etc.;

(d) the roof (contravariant) components $a^u = a^1$ and $a^v = a^2$ of the acceleration vector;

(e) the physical components $a^{(u)}$ and $a^{(v)}$ of the acceleration vector.

In addition, if $\mathbf{f} = uv\mathbf{g}_v$, write down the ***component form*** of Newton's Second Law in the $uv-$coordinate system.

3.(16 pts.) (a) Write the definitions in general coordinates of E^3 for $div\ A^i$ and $\nabla^2 I$.

(b) Express $div\ A^i$ and $\nabla^2 I$ in the cylindrical coordinate system $x^1 = r$, $x^2 = \theta$, $x^3 = z$. Recall that the Cartesian coordinates y^1, y^2, y^3 in E^3 are related to cylindrical coordinates by $y^1 = x^1 \cos(x^2)$, $y^2 = x^1 \sin(x^2)$, $y^3 = x^3$.

4.(16 pts.) (a) Show that the covariant derivatives of the metric and conjugate metric tensors are identically zero in any Riemannian space.

(b) Let V_N denote an arbitrary Riemannian space and g_{ij} its metric tensor. If B_i is a covariant vector in V_N, use part (a) and the definition of the covariant derivative to show that $\left(g^{ij}B_k\right)_{,m} = g^{ij}B_{k,m}$.

5.(16 pts.) Let y^1, y^2, y^3 denote Cartesian coordinates in E^3. Consider the surface $\mathcal{M}$ in E^3 given by

$$y^1 = x^1 \cos(x^2),\ \ y^2 = x^1 \sin(x^2),\ \ y^3 = x^1$$

where $0 < x^1 < \infty$ and $0 \le x^2 < 2\pi$.

(a) Show that the metric which $\mathcal{M}$ inherits from E^3 is $g_{11} = 2$, $g_{22} = \left(x^1\right)^2$, $g_{12} = g_{21} = 0$.

(b) Compute the nonvanishing Christoffel symbols of the second kind for $\mathcal{M}$.

(c) Write, BUT DO NOT SOLVE, the differential equations that define the geodesics in $\mathcal{M}$.

(d) Is the coordinate curve: $x^1 = s$, $x^2 =$ constant, a geodesic in $\mathscr{M}$? Why or why not?

6.(16 pts.) (a) Explain in detail how you would determine if a surface in E^3, given parametrically by

$$y^1 = f^1(x^1, x^2),\ \ y^2 = f^2(x^1, x^2),\ \ y^3 = f^3(x^1, x^2)$$

is flat.

(b) Determine whether or not the surface $\mathscr{M}$ in problem 5 is flat.