Mathematics 322 | Final Exam | Name:
Vector and Tensor Analysis | Winter 1999

Work any six of the following seven problems. Each problem has the same point value (25 points). Circle the numbers of the problems you wish to have graded.

1. In cylindrical coordinates r, θ, z in three-dimensional Euclidean space, a vector field is such that the vector at each point points along the coordinate curve of θ, in the sense of increasing θ, and has magnitude $3r$. Find the contravariant and covariant components of this vector field in the cylindrical coordinate system and in the rectangular Cartesian coordinate system.

2. Let y^1, y^2, y^3 denote rectangular Cartesian coordinates in three-dimensional Euclidean space. Consider the surface $\mathfrak{M}$ described parametrically by

$$y^1 = x^1 \cos(x^2)$$
$$y^2 = x^1 \sin(x^2)$$
$$y^3 = x^1$$

where $x^1 > 0$ and $0 \leq x^2 < 2\pi$.

(a) Sketch the surface and show several typical x^1 and x^2 coordinate curves in the surface.
(b) Find the metric tensor for $\mathfrak{M}$.
(c) Find all nonvanishing Christoffel symbols for $\mathfrak{M}$.
(d) Find the geodesics for $\mathfrak{M}$.

5 bonus points: (e) Find the distance from the point $(x^1, x^2) = (1, 0)$ to $(x^1, x^2) = (2, \pi/2)$ in $\mathfrak{M}$.

3. Let θ denote the angle between the x^1 and x^2 coordinate curves at a point P in a positive-definite Riemannian space. Show that

$$\cos(\theta) = \frac{a_{12}}{\sqrt{a_{11} a_{22}}} .$$

4. Show that the covariant derivatives of the metric and conjugate metric tensors are identically zero in any Riemannian space.

5. Let T^r and S_r be vector fields along a curve C in a Riemannian space.

(a) If the absolute derivatives of T^r and S_r are zero along C, show that the invariant $T^r S_r$ is constant along C.

(b) If the absolute derivative of T^r along C is zero and the invariant $T^r S_r$ is constant along C, what can be said about S_r?

6. Let ψ be a scalar invariant and let T^s be a tensor in a positive-definite Riemannian space $\mathfrak{M}$. Let a denote the determinant of the metric tensor a_{mn} in $\mathfrak{M}$.

(a) Show that $T^{i}{}_{|i} = \frac{1}{\sqrt{a}} \frac{\partial}{\partial x^{i}}(\sqrt{a}\, T^{i})$.

(b) Write out explicitly (the right-hand side of) the formula for the invariant $T^{i}{}_{|i}$ in part (a) for spherical coordinates r, θ, ϕ in three-dimensional Euclidean space.

(c) Use part (a) and problem 4 to show that

$$a^{mn}\psi_{|mn} = \frac{1}{\sqrt{a}} \frac{\partial}{\partial x^{i}}\left(\sqrt{a}\, a^{ij} \frac{\partial \psi}{\partial x^{j}}\right).$$

(d) Write out explicitly (the right-hand side of) the invariant $a^{mn}\psi_{|mn}$ in part (c) for spherical coordinates r, θ, ϕ in three-dimensional Euclidean space.

7. Let $\mathfrak{M}$ be a Riemannian space with metric tensor a_{mn} and dimension 4.

(a) If the quantities $R(p,r,m,n)$ satisfy the relations

$$T_{r|mn} - T_{r|nm} = R(p,r,m,n)\, T^{p}$$

for an arbitrary contravariant vector T^{s} on $\mathfrak{M}$, show that the quantities $R(p,r,m,n)$ are the components of a tensor on $\mathfrak{M}$. Determine the rank and covariant and contravariant types of this tensor.

(b) Show that the tensor of part (a) has the following symmetry properties:

$$R(p,r,s,t) = -R(r,p,s,t)$$

$$R(p,r,s,t) = -R(p,r,t,s)$$

$$R(p,r,s,t) = R(s,t,p,r)$$

$$R(p,r,s,t) + R(p,s,t,r) + R(p,t,r,s) = 0.$$

(c) Use part (b) to show that the tensor of part (a) has 6 distinct nonvanishing components of type $R(p,r,p,r)$, 12 of type $R(p,r,p,t)$, and 2 of type $R(p,r,s,t)$.

(d) How many distinct nonvanishing components does the tensor in part (a) have?