Mathematics 5222 Final Exam Name:____________________
Spring 2015

This is an open-book examination; i.e. you may refer at any time to your textbook, "Tensor Analysis" by Sokolnikoff. You will have 180 minutes to complete your solutions to the problems on this exam.

1.(40 pts.) Write the transformation law for the components of the following under admissible coordinate transformations.

(a) a covariant vector; (b) a contravariant vector; (c) a mixed tensor of rank two.

(d) Write out explicitly the laws of transformation for the components of a contravariant vector in two-dimensional euclidean space when S is the transformation from polar coordinates x^1, x^2 to rectangular cartesian coordinates y^1, y^2 given by

$$S: \quad \begin{cases} y^1 = x^1 \cos\left(x^2\right) \\ y^2 = x^1 \sin\left(x^2\right) \end{cases}$$

where $x^1 > 0$ and $0 \le x^2 < 2\pi$.

2.(40 pts.) (a) If all the components of a tensor vanish in one coordinate system, show that they necessarily vanish in all other coordinate systems.

(b) Show that a rank two tensor whose components are skew-symmetric in one coordinate system necessarily has skew-symmetric components in any coordinate system.

(c) How many independent components are there in a skew-symmetric tensor of rank two? Justify your answer in a general $N-$dimensional Riemannian space.

(d) If b_{ij} are the covariant components of a skew-symmetric tensor and A^i are the contravariant components of a vector, show that $b_{ij}A^iA^j=0$.

3.(40 pts.) Let V_N be a Riemannian space of dimension N with covariant metric tensor a_{ij}. Define the divergence of a vector field A on V_N with (continuously differentiable) contravariant components A^i to be the scalar invariant $div\left(A\right) = A^i_{,i}$. Define the gradient of a (continuously differentiable) scalar invariant u on V_N to be the rank one tensor $grad\left(u\right)$ with covariant components $grad\left(u\right)_i = \dfrac{\partial u}{\partial x^i}$. Define the Laplacian of a (twice continuously differentiable) scalar invariant φ on V_N to be the scalar invariant $\Delta\left(\varphi\right) = div\left(grad\left(\varphi\right)\right)$.

(a) Show that Laplacian of φ can be expressed as $\Delta\left(\varphi\right) = a^{ij}\varphi_{,ij}$.

(b) Show that $\Delta\left(\varphi\right) = \dfrac{1}{\sqrt{a}}\dfrac{\partial}{\partial x^i}\left(\sqrt{a}a^{ij}\dfrac{\partial\varphi}{\partial x^j}\right)$.

(b) Show that the general definitions of gradient, divergence, and the Laplacian reduce to the usual definitions in the rectangular cartesian coordinate system $y^1,\ldots,y^N$ in a euclidean space of dimension N.

(c) Write out explicitly the gradient, divergence, and Laplacian in spherical coordinates in three-dimensional euclidean space. (Recall that the cartesian coordinates in $\mathbb{E}_3$ are related to spherical coordinates by the transformation formulas $y^1 = x^1\cos\left(x^2\right)\sin\left(x^3\right)$, $y^2 = x^1\sin\left(x^2\right)\sin\left(x^3\right)$, $y^3 = x^1\cos\left(x^3\right)$.)

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Problems 4 and 5 will be concerned with the hyperbolic disk, V_2, consisting of the points interior to the disk of radius 2 and center at the origin in the two-dimensional euclidean plane, given parametrically by

$$y^1 = x^1 \cos\left(x^2\right), \quad y^2 = x^1 \sin\left(x^2\right)$$

where $0 \le x^1 < 2$ and $0 \le x^2 < 2\pi$. Equip V_2 with the metric tensor

$$a_{11} = \frac{1}{\left\{1 - \frac{\left(x^1\right)^2}{4}\right\}^2}, \quad a_{22} = \frac{\left(x^1\right)^2}{\left\{1 - \frac{\left(x^1\right)^2}{4}\right\}^2}, \quad a_{12} = a_{21} = 0.$$

4.(40 pts.) (a) Compute the 16 covariant components of the Riemann-Christoffel curvature tensor R_{ijkl} for the hyperbolic disk. (Hint: Why does it suffice to compute R_{1212}?)

(b) Show that the Gaussian curvature

$$\kappa = \frac{R_{1212}}{a}$$

of the hyperbolic disk is given by $\kappa = -1$.

5.(40 pts.) Show that the geodesics for the hyperbolic disk are either straight lines through the origin or circles which meet the boundary of the disk orthogonally by carrying out the following steps.

(a) Show that geodesics for the hyperbolic disk satisfy the following system of differential equations:

$$\frac{d^2x^1}{ds^2} + \frac{\left(x^1/2\right)}{\left(1-\left(x^1\right)^2/4\right)}\left(\frac{dx^1}{ds}\right)^2 - \frac{x^1\left(1+\left(x^1\right)^2/4\right)}{\left(1-\left(x^1\right)^2/4\right)}\left(\frac{dx^2}{ds}\right)^2 = 0$$

$$\frac{d^2x^2}{ds^2} + \frac{2\left(1+\left(x^1\right)^2/4\right)}{x^1\left(1-\left(x^1\right)^2/4\right)}\left(\frac{dx^1}{ds}\right)\left(\frac{dx^2}{ds}\right) = 0$$

(b) Show that $p\left(x^1\right) = \dfrac{\left(x^1\right)^2}{\left(1-\left(x^1\right)^2/4\right)^2}$ is an integrating factor for the second differential equation in part (a).

(c) Multiply the second differential equation in part (a) by the integrating factor in part (b) and then integrate once to obtain

$$\frac{\left(x^1\right)^2}{\left(1-\left(x^1\right)^2/4\right)^2}\frac{dx^2}{ds} = c_1.$$

(d) If the constant $c_1 = 0$ in part (c), what is the resulting geodesic?

(e) If $c_1 \neq 0$, substitute from part (c) into the identity $1 = a_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}$ and rearrange to obtain

$$\frac{dx^1}{ds} = \pm\frac{\left(1-\left(x^1\right)^2/4\right)\left(\left(x^1\right)^2 - c_1^2\left(1-\left(x^1\right)^2/4\right)^2\right)^{1/2}}{x^1}.$$

(f) Use the chain rule $\dfrac{dx^2}{dx^1} = \dfrac{dx^2/ds}{dx^1/ds}$ and parts (c) and (e) to obtain an integral expression for x^2 in terms of x^1 and evaluate the integral to obtain $\cos\left(x^2 - c_2\right) = c_3\left(1+\left(x^1\right)^2/4\right)\Big/x^1$. Etc.