Math 5222

Homework Set #1

$\boxed{\#1.5}$ Using the vectors $\vec{u} \sim (2,-3,4)$ and $\vec{v} \sim (1,0,1)$ given in Exercise 1.4, compute $\mathrm{Proj}_{\vec{u}}(\vec{v})$ and $\mathrm{Proj}_{\vec{v}}(\vec{u})$.

$$\mathrm{Proj}_{\vec{u}}(\vec{v}) = (\vec{v}\cdot\hat{u})\hat{u} \sim \left((1,0,1)\cdot\frac{1}{\sqrt{29}}(2,-3,4)\right)\frac{1}{\sqrt{29}}(2,-3,4)$$

$$|\vec{u}| = \sqrt{2^2+(-3)^2+4^2} = \sqrt{29}$$

$$= \frac{6}{\sqrt{29}}\cdot\frac{1}{\sqrt{29}}(2,-3,4)$$

$$= \boxed{\frac{6}{29}(2,-3,4)}$$

$$\mathrm{Proj}_{\vec{v}}(\vec{u}) = (\vec{u}\cdot\hat{v})\hat{v} \sim \left((2,-3,4)\cdot\frac{1}{\sqrt{2}}(1,0,1)\right)\frac{1}{\sqrt{2}}(1,0,1)$$

$$|\vec{v}| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$$

$$= \frac{6}{2}(1,0,1)$$

$$= \boxed{(3,0,3)}$$

$\boxed{\#1.7}$ Using

$$(1.11) \qquad \vec{u}\cdot\vec{v} = \frac{1}{2}\left(|\vec{u}|^2 + |\vec{v}|^2 - |\vec{v}-\vec{u}|^2\right),$$

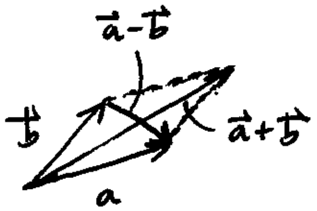
prove the distributive law

$$(1.14) \qquad \vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} \qquad \text{for all } \vec{u},\vec{v},\vec{w},$$

without introducing Cartesian coordinates.

We will use the fact (from plane geometry) that if $\vec{a}$ and $\vec{b}$ are the co-terminal edges of a parallelogram, then

$$(*) \qquad 2|\vec{a}|^2 + 2|\vec{b}|^2 = |\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2. \qquad \text{(Parallelogram Law... see digram)}$$



Let $\vec{u}, \vec{v}$, and $\vec{w}$ be arbitrary vectors in $\mathbb{E}^3$. Then

$$\vec{u}\cdot(\vec{v}+\vec{w}) = \frac{1}{2}\left(|\vec{u}|^2 + |\vec{v}+\vec{w}|^2 - |\vec{v}+\vec{w}-\vec{u}|^2\right) \qquad \text{by } (1.11)$$

$$= \frac{1}{2}\left(|\vec{u}|^2 + |\vec{v}+\vec{w}|^2 - |(\vec{v}-\tfrac{1}{2}\vec{u}) + (\vec{w}-\tfrac{1}{2}\vec{u})|^2\right)$$

$$= \frac{1}{2}\left(|\vec{u}|^2 + |\vec{v}+\vec{w}|^2 - 2|\vec{v}-\tfrac{1}{2}\vec{u}|^2 - 2|\vec{w}-\tfrac{1}{2}\vec{u}|^2 + |(\vec{v}-\cancel{\tfrac{1}{2}\vec{u}}) - (\vec{w}-\cancel{\tfrac{1}{2}\vec{u}})|^2\right) \qquad \text{by } (*)$$

$$= \frac{1}{2}\left(|\vec{u}|^2 + |\vec{v}+\vec{w}|^2 + |\vec{v}-\vec{w}|^2 - 2|\tfrac{1}{2}\vec{v} + (\tfrac{1}{2}\vec{v}-\tfrac{1}{2}\vec{u})|^2 - 2|\tfrac{1}{2}\vec{w} + (\tfrac{1}{2}\vec{w}-\tfrac{1}{2}\vec{u})|^2\right)$$

$$= \frac{1}{2}\Big[|\vec{u}|^2 + 2|\vec{v}|^2 + 2|\vec{w}|^2 - 2\left(2|\tfrac{1}{2}\vec{v}|^2 + 2|\tfrac{1}{2}(\vec{v}-\vec{u})|^2 - |\cancel{\tfrac{1}{2}\vec{v}} - (\cancel{\tfrac{1}{2}\vec{v}} - \tfrac{1}{2}\vec{u})|^2\right)$$

$$-2\left(2|\tfrac{1}{2}\vec{w}|^2 + 2|\tfrac{1}{2}(\vec{w}-\vec{u})|^2 - |\cancel{\tfrac{1}{2}\vec{w}} - (\cancel{\tfrac{1}{2}\vec{w}} - \tfrac{1}{2}\vec{u})|^2\right)\Big] \qquad \text{by } (*).$$

$$\therefore\ \vec{u}\cdot(\vec{v}+\vec{w}) = \frac{1}{2}\Big[|\vec{u}|^2 + 2|\vec{v}|^2 + 2|\vec{w}|^2 - |\vec{v}|^2 - |\vec{v}-\vec{u}|^2 + \frac{1}{2}|\vec{u}|^2$$

$$-|\vec{w}|^2 - |\vec{w}-\vec{u}|^2 + \frac{1}{2}|\vec{u}|^2\Big]$$

$$= \frac{1}{2}\Big[|\vec{u}|^2 + |\vec{v}|^2 - |\vec{v}-\vec{u}|^2\Big] + \frac{1}{2}\Big[|\vec{u}|^2 + |\vec{w}|^2 - |\vec{w}-\vec{u}|^2\Big]$$

$$= \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} \qquad \text{by (1.11).}$$

$\boxed{\#1.12}$ (We will use the notation $\vec{u}\otimes\vec{v}$ instead of $\vec{u}\vec{v}$ for the tensor (or direct) product of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{E}^3$. That is,

$$(\vec{u}\otimes\vec{v})(\vec{w}) = \vec{u}(\vec{v}\cdot\vec{w}) \quad \text{for all } \vec{w} \text{ in } \mathbb{E}^3.)$$

If $\vec{u}\sim(1,-1,2)$, $\vec{v}\sim(3,2,1)$, and $\vec{w}\sim(4,1,7)$, compute

(a) $(\vec{u}\otimes\vec{v})(\vec{w})$ (b) $(\vec{v}\otimes\vec{u})(\vec{w})$ (c) $(\vec{w}\otimes\vec{u})(\vec{u})$.

(a) $$(\vec{u}\otimes\vec{v})(\vec{w}) = \vec{u}(\vec{v}\cdot\vec{w}) \sim (1,-1,2)((3,2,1)\cdot(4,1,7))$$
$$= (3\cdot4+2\cdot1+1\cdot7)(1,-1,2)$$
$$= \boxed{21(1,-1,2)} \quad \text{or} \quad (21,-21,42).$$

(b) $$(\vec{v}\otimes\vec{u})(\vec{w}) = \vec{v}(\vec{u}\cdot\vec{w}) \sim (3,2,1)((1,-1,2)\cdot(4,1,7))$$
$$= (1\cdot4+(-1)\cdot1+2\cdot7)(3,2,1)$$
$$= \boxed{17(3,2,1)} \quad \text{or} \quad (51,34,17).$$

Note: The results of (a) and (b) show that $\vec{u}\otimes\vec{v} \neq \vec{v}\otimes\vec{u}$ in general. Therefore tensor products are not "commutative".

(c) $$(\vec{w}\otimes\vec{u})(\vec{u}) = \vec{w}(\vec{u}\cdot\vec{u}) \sim (4,1,7)((1,-1,2)\cdot(1,-1,2))$$
$$= (1\cdot1+(-1)(-1)+2\cdot2)(4,1,7)$$
$$= \boxed{6(4,1,7)} \quad \text{or} \quad (24,6,42).$$

$\boxed{\#1.18}$ Let A be an arbitrary, 3-dimensional skew tensor.

(a) By expressing A in terms of its Cartesian components (and noting that only 3 of these can be assigned arbitrarily since $A = -A^T$), find a vector $\vec{\omega}$ such that $A\vec{v} = \vec{\omega} \times \vec{v}$ for all $\vec{v}$.

(b) Use the results of Exercise 1.16 to show that $\vec{\omega}$ is unique. $\vec{\omega}$ is called the axis of A and is important in rigid body dynamics. See Exercise 4.20.

(c) Show that $\vec{v} \cdot A\vec{v} = 0$ for all $\vec{v}$, in any number of dimensions.

(d) Show that $A\vec{\omega} = \vec{0}$. Check this result by using the numerical values obtained in Exercise 1.17(c).

(a) Suppose $A \sim \begin{bmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{bmatrix}$. Then $A^T \sim \begin{bmatrix} A_{xx} & A_{yx} & A_{zx} \\ A_{xy} & A_{yy} & A_{zy} \\ A_{xz} & A_{yz} & A_{zz} \end{bmatrix}$

so $A = -A^T$ if and only if $\begin{cases} A_{xx} = A_{yy} = A_{zz} = 0, \\ A_{yx} = -A_{xy},\ A_{zx} = -A_{xz},\ A_{zy} = -A_{yz}. \end{cases}$

Therefore $A \sim \begin{bmatrix} 0 & A_{xy} & A_{xz} \\ -A_{xy} & 0 & A_{yz} \\ -A_{xz} & -A_{yz} & 0 \end{bmatrix}$ where A_{xy}, A_{xz}, and A_{yz} are arbitrary real numbers. By Problem 1.6, pp. 18-19, the Cartesian components of $\vec{u}\times$ are given by

$$\vec{u}\times \sim \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}.$$

Comparing Cartesian components of A and $\vec{u}\times$ we see that

40% if $\boxed{\vec{\omega} \sim (-A_{yz}, A_{xz}, -A_{xy})}$ then $A = \vec{\omega}\times$; ie $A\vec{v} = \vec{\omega}\times\vec{v}$

for all vectors $\vec{v}$ in $\mathbb{E}^3$.

(b) Suppose $A\vec{v} = \vec{\omega}\times\vec{v}$ and $A\vec{v} = \vec{\sigma}\times\vec{v}$ for all $\vec{v}$ in $\mathbb{E}^3$. Then

10% $\vec{\omega}\times\vec{v} = \vec{\sigma}\times\vec{v}$ for all $\vec{v}$ in $\mathbb{E}^3$, so by Exercise 1.16, $\vec{\omega} = \vec{\sigma}$; i.e. $\vec{\omega}$ is unique.

(c) Since $A = -A^T$, we have

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$$-A\vec{v}\cdot\vec{v} = \vec{v}\cdot(-Av) = \vec{v}\cdot A^T\vec{v} \overset{(1.33)}{=} A\vec{v}\cdot\vec{v}$$

for all $\vec{v}$ in $\mathbb{E}^n$. That is $2A\vec{v}\cdot\vec{v} = 0$ so $A\vec{v}\cdot\vec{v} = 0$.

(d) $A\vec{\omega} \overset{\text{part(a)}}{=} \vec{\omega}\times\vec{\omega} = \vec{0}$. From Exercise 1.17,

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$$A \sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -3/2 \\ -1 & 3/2 & 0 \end{bmatrix} \quad \text{and} \quad \vec{\omega} \sim \left(\tfrac{3}{2}, 1, 0\right).$$

Therefore $A\vec{\omega} \sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -3/2 \\ -1 & 3/2 & 0 \end{bmatrix}\begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $A\vec{\omega} = \vec{0}$.

$\boxed{\#1.20}$ As in Figure 1.10, let $\hat{n}\,dA$ denote an oriented differential element of area at a point P and time t in a continuum (e.g. a fluid or solid) and let $\vec{t}\,dA$ denote the force that the material into which $\vec{n}$ points exerts across dA. $\vec{t}$ is called the stress at P and t in the direction $\hat{n}$, $\vec{t}_n \equiv \mathrm{Proj}_{\vec{n}}(\vec{t})$ the normal stress, and $\vec{t}_s \equiv \vec{t} - \vec{t}_n$ the shear stress. By considering the equations of motion of a tetrahedron of the material of arbitrarily small volume, instantaneously centered at P, it can be shown that $\vec{t} = T\hat{n}$, where $T = T^T$ is the (Cauchy) stress tensor at P and t. If

$$T \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \quad \text{and} \quad \vec{n} \sim (1,2,-1),$$

compute the normal and shear stress.

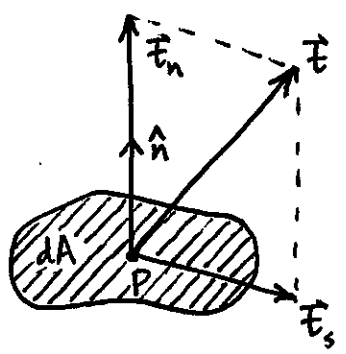

Figure 1.10

Stress: $\vec{t} = T\hat{n} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 7 \\ -5 \end{bmatrix}.$

$|\vec{n}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$

Normal Stress: $\vec{t}_n = \mathrm{Proj}_{\vec{n}}(\vec{t}) = (\vec{t}\cdot\hat{n})\hat{n} \sim \left(\frac{1}{\sqrt{6}}(1,7,-5)\cdot\frac{1}{\sqrt{6}}(1,2,-1)\right)\frac{1}{\sqrt{6}}(1,2,-1)$

$$\therefore \boxed{\vec{t}_n \sim \frac{10}{3\sqrt{6}}(1,2,-1)}.$$

Shear Stress: $\vec{t}_s = \vec{t} - \vec{t}_n \sim \frac{1}{\sqrt{6}}(1,7,-5) - \frac{10}{3\sqrt{6}}(1,2,-1)$

$$\boxed{\vec{t}_s \sim \frac{1}{\sqrt{6}}\left(-\frac{7}{3}, \frac{1}{3}, \frac{-5}{3}\right)} \sim \frac{1}{3\sqrt{6}}(-7,1,-5)$$