Math 5222

Homework Set #2

#2.2 Let $\vec{g}_1 \sim (-1,0,0)$, $\vec{g}_2 \sim (1,1,0)$, $\vec{g}_3 \sim (1,1,1)$, and $\vec{v} \sim (1,2,3)$.
Compute the reciprocal base vectors and the contravariant and covariant components of $\vec{v}$.

$$G = \left[\vec{g}_1\ \vec{g}_2\ \vec{g}_3\right] = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\left[G \,\vdots\, I\right] \xrightarrow[-R_3+R_1]{-R_3+R_2} \left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1]{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = \left[I \,\vdots\, G^{-1}\right]$$

The rows of G^{-1} are the Cartesian components of the vectors $\vec{g}^1, \vec{g}^2, \vec{g}^3$ in the
40% reciprocal base. Therefore $\boxed{\vec{g}^1 \sim (-1,1,0),\ \vec{g}^2 \sim (0,1,-1),\ \vec{g}^3 \sim (0,0,1)}$.

The contravariant components of $\vec{v}$ are:

$$v^1 = \vec{g}^1 \cdot \vec{v} = (-1,1,0)\cdot(1,2,3) = \boxed{1},$$

$$v^2 = \vec{g}^2 \cdot \vec{v} = (0,1,-1)\cdot(1,2,3) = \boxed{-1},$$

30% $$v^3 = \vec{g}^3 \cdot \vec{v} = (0,0,1)\cdot(1,2,3) = \boxed{3}.$$

The covariant components of $\vec{v}$ are:

$$v_1 = \vec{g}_1 \cdot \vec{v} = (-1,0,0)\cdot(1,2,3) = \boxed{-1},$$

30% $$v_2 = \vec{g}_2 \cdot \vec{v} = (1,1,0)\cdot(1,2,3) = \boxed{3},$$

$$v_3 = \vec{g}_3 \cdot \vec{v} = (1,1,1)\cdot(1,2,3) = \boxed{6}.$$

#2.5 Establish (2.17): $\epsilon^{ijk}\epsilon_{pqr} = \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ \delta^k_p & \delta^k_q & \delta^k_r \end{vmatrix}$.

30% <u>Case 1</u>: There is a repeated index among i,j,k or among p,q,r.

In this case, either $\epsilon^{ijk} = 0$ by (2.16) or $\epsilon_{pqr} = 0$ by (2.13), so $\epsilon^{ijk}\epsilon_{pqr} = 0$. Also, if there is a repeated index among i,j,k then two rows are identical in the determinant on the right side of (2.17) and thus its value is zero. Likewise, if there is a repeated index among p,q,r then two columns in the determinant are identical, so again the right side of (2.17) is zero.

70% <u>Case 2</u>: The indices i,j,k are distinct and the indices p,q,r are distinct.

In this case there are permutations σ and τ of $(1,2,3)$ such that $\sigma(1,2,3) = (i,j,k)$ and $\tau(1,2,3) = (p,q,r)$. We define

$$(-1)^\sigma = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation,} \\ -1 & \text{if } \sigma \text{ is an odd permutation,} \end{cases}$$

and similarly for $(-1)^\tau$. By (2.13) and (2.16),

$$\epsilon^{ijk}\epsilon_{pqr} = (-1)^\sigma J^{-1} \cdot (-1)^\tau J = (-1)^\sigma(-1)^\tau .$$

On the other hand, by interchanging rows we find

$$(*) \qquad \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ \delta^k_p & \delta^k_q & \delta^k_r \end{vmatrix} = (-1)^\sigma \begin{vmatrix} \delta^1_p & \delta^1_q & \delta^1_r \\ \delta^2_p & \delta^2_q & \delta^2_r \\ \delta^3_p & \delta^3_q & \delta^3_r \end{vmatrix} .$$

By interchanging columns we find

$$(**) \qquad \begin{vmatrix} \delta^1_p & \delta^1_q & \delta^1_r \\ \delta^2_p & \delta^2_q & \delta^2_r \\ \delta^3_p & \delta^3_q & \delta^3_r \end{vmatrix} = (-1)^\tau \begin{vmatrix} \delta^1_1 & \delta^1_2 & \delta^1_3 \\ \delta^2_1 & \delta^2_2 & \delta^2_3 \\ \delta^3_1 & \delta^3_2 & \delta^3_3 \end{vmatrix} = (-1)^\tau \det(I) = (-1)^\tau .$$

Combining $(*)$ and $(**)$, we see that the right side of (2.17) is $(-1)^\sigma \cdot (-1)^\tau$. This establishes the identity (2.17).

#2.6 (a) Establish (2.18): $\epsilon^{ijk}\epsilon_{pqk} = \delta^i_p\delta^j_q - \delta^i_q\delta^j_p$

by expanding the determinant in (2.17) by, say, its first row and then setting $r=k$.

(b) Use (2.18) to show that $\epsilon^{ijk}\epsilon_{pjk} = 2\delta^i_p$.

By expanding the determinant using cofactors along row 1

(2.17) $$\epsilon^{ijk}\epsilon_{pqr} = \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ \delta^k_p & \delta^k_q & \delta^k_r \end{vmatrix}$$

gives

30% to here.

(*) $$\epsilon^{ijk}\epsilon_{pqr} = \delta^i_p(\delta^j_q\delta^k_r - \delta^k_q\delta^j_r) - \delta^i_q(\delta^j_p\delta^k_r - \delta^k_p\delta^j_r) + \delta^i_r(\delta^j_p\delta^k_q - \delta^k_p\delta^j_q).$$

Setting $r=k$ in (*) and then summing from $k=1$ to 3 gives

(2.18) $$\begin{aligned}\epsilon^{ijk}\epsilon_{pqk} &= \delta^i_p\delta^j_q\delta^k_k - \delta^i_p\delta^k_q\delta^j_k - \delta^i_q\delta^j_p\delta^k_k + \delta^i_q\delta^k_p\delta^j_k + \delta^i_k\delta^j_p\delta^k_q - \delta^i_k\delta^k_p\delta^j_q \\ &= 3\delta^i_p\delta^j_q - \delta^i_p\delta^j_q - 3\delta^i_q\delta^j_p + \delta^i_q\delta^j_p + \delta^i_q\delta^j_p - \delta^i_p\delta^j_q \\ &= \delta^i_p\delta^j_q - \delta^i_q\delta^j_p.\end{aligned}$$

70% to here.

Setting $q=j$ in (2.18) and then summing from $j=1$ to 3 gives

$$\epsilon^{ijk}\epsilon_{pjk} = \delta^i_p\delta^j_j - \delta^i_j\delta^j_p = 3\delta^i_p - \delta^i_p = 2\delta^i_p.$$

100% to here.

#2.7 Establish the vector triple product identity

(1.22) $$(\vec{u}\times\vec{v})\times\vec{w} = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{v}\cdot\vec{w})\vec{u}$$

by first computing the contravariant components of $\vec{u}\times\vec{v}$ and then the covariant components of $(\vec{u}\times\vec{v})\times\vec{w}$. Finally, use the identity

(2.18) $$\epsilon^{ijk}\epsilon_{pqk} = \delta^i_p\delta^j_q - \delta^i_q\delta^j_p \quad .$$

The contravariant components of $\vec{u}\times\vec{v}$ are :

$$\begin{aligned}(\vec{u}\times\vec{v})^k &= (\vec{u}\times\vec{v})\cdot\vec{g}^k \\ &= (u_i\vec{g}^i \times v_j\vec{g}^j)\cdot\vec{g}^k \\ &= u_i v_j(\vec{g}^i\times\vec{g}^j)\cdot\vec{g}^k \\ &= u_i v_j \epsilon^{ijk} .\end{aligned}$$

The covariant components of $(\vec{u}\times\vec{v})\times\vec{w}$ are :

$$\begin{aligned}\left[(\vec{u}\times\vec{v})\times\vec{w}\right]_m &= \left[(\vec{u}\times\vec{v})\times\vec{w}\right]\cdot\vec{g}_m \\ &= \left[((\vec{u}\times\vec{v})^k\vec{g}_k)\times(w^\ell\vec{g}_\ell)\right]\cdot\vec{g}_m \\ &= \left[(\vec{u}\times\vec{v})^k w^\ell\right](\vec{g}_k\times\vec{g}_\ell)\cdot\vec{g}_m \\ &= \left[u_i v_j \epsilon^{ijk} w^\ell\right](\vec{g}_\ell\times\vec{g}_m)\cdot\vec{g}_k \\ &= (u_i v_j w^\ell)\epsilon^{ijk}\epsilon_{\ell mk} .\end{aligned}$$

Applying (2.18) we find

$$\left[(\vec{u}\times\vec{v})\times\vec{w}\right]_m = (u_i v_j w^\ell)(\delta^i_\ell \delta^j_m - \delta^i_m \delta^j_\ell)$$

$$= u_i \delta^i_\ell w^\ell v_j \delta^j_m - v_j \delta^j_\ell w^\ell u_i \delta^i_m$$

$$= (u_\ell w^\ell) v_m - (v_\ell w^\ell) u_m .$$

Consequently,

$$(\vec{u}\times\vec{v})\times\vec{w} = \left[(\vec{u}\times\vec{v})\times\vec{w}\right]_m \vec{g}^{\,m}$$

$$= \left[(u_\ell w^\ell) v_m - (v_\ell w^\ell) u_m\right]\vec{g}^{\,m}$$

$$= (u_\ell w^\ell) v_m \vec{g}^{\,m} - (v_\ell w^\ell) u_m \vec{g}^{\,m}$$

$$= (\vec{u}\cdot\vec{w})\vec{v} - (\vec{v}\cdot\vec{w})\vec{u} .$$

\#2.9 Raising and lowering of indices. Show that

$$\vec{g}^i = g^{ik}\vec{g}_k\,,\quad \vec{g}_i = g_{ik}\vec{g}^k,\quad v^i = g^{ik}v_k\,,\quad v_i = g_{ik}v^k,$$

$$T^i_{\cdot j} = g^{ik}T_{kj}\,,\quad T^{ij} = g^{ik}T_k^{\cdot j}\,,\quad T_{ij} = g_{ik}T^k_{\cdot j}\,,\quad \text{etc.}$$

25% for each verification (up to 100%)

-10% for each of the three categories if missing

① $$g^{ik}\vec{g}_k = (\vec{g}^i\cdot\vec{g}^k)\vec{g}_k \overset{\text{by (1.29)}}{=} (\vec{g}_k\vec{g}^k)(\vec{g}^i) \overset{\text{by ex. 2.8}}{=} \mathbb{1}(\vec{g}^i) = \vec{g}^i$$

② $$g_{ik}\vec{g}^k = (\vec{g}_i\cdot\vec{g}_k)\vec{g}^k \overset{\text{by (1.29)}}{=} (\vec{g}^k\vec{g}_k)(\vec{g}_i) \overset{\text{by ex. 2.8}}{=} \mathbb{1}(\vec{g}_i) = \vec{g}_i$$

③ $$g^{ik}v_k = (\vec{g}^i\cdot\vec{g}^k)v_k = \vec{g}^i\cdot(v_k\vec{g}^k) \overset{\text{by (2.5)}}{=} \vec{g}^i\cdot\vec{v} \overset{\text{by (2.7)}}{=} v^i$$

④ $$g_{ik}v^k = (\vec{g}_i\cdot\vec{g}_k)v^k = \vec{g}_i\cdot(v^k\vec{g}_k) \overset{\text{by (2.1)}}{=} \vec{g}_i\cdot\vec{v} \overset{\text{by (2.6)}}{=} v_i$$

⑤ $$g^{ik}T_{kj} \overset{\text{by (2.21)}}{=} g^{ik}(\vec{g}_k\cdot T\vec{g}_j) = (g^{ik}\vec{g}_k)\cdot T\vec{g}_j \overset{\text{by ①}}{=} \vec{g}^i\cdot T\vec{g}_j \overset{\text{by (2.27)}}{=} T^i_{\cdot j}$$

⑥ $$g^{ik}T_k^{\cdot j} \overset{\text{by (2.28)}}{=} g^{ik}(\vec{g}_k\cdot T\vec{g}^j) = (g^{ik}\vec{g}_k)\cdot T\vec{g}^j \overset{\text{by ①}}{=} \vec{g}^i\cdot T\vec{g}^j \overset{\text{by (2.26)}}{=} T^{ij}$$

⑦ $$g_{ik}T^k_{\cdot j} \overset{\text{by (2.27)}}{=} g_{ik}(\vec{g}^k\cdot T\vec{g}_j) = (g_{ik}\vec{g}^k)\cdot T\vec{g}_j \overset{\text{by ②}}{=} \vec{g}_i\cdot Tg_j \overset{\text{by (2.21)}}{=} T_{ij}$$

\# 2.10 Noting that $[g_{ij}] = G^T G$, show that

(a) $\det[g_{ij}] = J^2$,

(b) $g^{ik} g_{kj} = \delta^i_j$.

Note: $$G^T G = \begin{bmatrix} \vec{g}_1^{\,T} \\ \vec{g}_2^{\,T} \\ \vdots \\ \vec{g}_n^{\,T} \end{bmatrix} \begin{bmatrix} \vec{g}_1 \vec{g}_2 \cdots \vec{g}_n \end{bmatrix} = \begin{bmatrix} \vec{g}_1\cdot\vec{g}_1 & \vec{g}_1\cdot\vec{g}_2 & \cdots & \vec{g}_1\cdot\vec{g}_n \\ \vec{g}_2\cdot\vec{g}_1 & \vec{g}_2\cdot\vec{g}_2 & \cdots & \vec{g}_2\cdot\vec{g}_n \\ \vdots & \vdots & & \\ \vec{g}_n\cdot\vec{g}_1 & \vec{g}_n\cdot\vec{g}_2 & \cdots & \vec{g}_n\cdot\vec{g}_n \end{bmatrix} = [g_{ij}].$$

(a) Because $[g_{ij}] = G^T G$ and $\det(AB) = \det(A)\det(B)$, $\det(A^T) = \det A$, we have

$$\det[g_{ij}] = \det(G^T G) = \det(G^T)\det(G) = [\det(G)]^2 = J^2.$$

(b) $$g^{ik} g_{kj} = (\vec{g}^i \cdot \vec{g}^k)(\vec{g}_k \cdot \vec{g}_j) = ((\vec{g}^i \cdot \vec{g}^k)\vec{g}_k)\cdot \vec{g}_j \overset{\text{by (1.29)}}{=} (\vec{g}_k g^k(\vec{g}^i))\cdot \vec{g}_j$$

$$\overset{\text{by ex. 2.8}}{=} (\mathbb{1}(\vec{g}^i))\cdot \vec{g}_j = \vec{g}^i \cdot \vec{g}_j = \delta^i_j.$$

#2.20 Find the 2×2 matrix of the Cartesian components of the rotator that sends $\vec{e}_x$ into the unit vector $\vec{e}_x \cos(\theta) + \vec{e}_y \sin(\theta)$.

By exercise 2.16, a rotator $\mathbb{Q}$ in $\mathbb{E}^n$ is a second order tensor on $\mathbb{E}^n$ which is orthogonal (i.e. $\mathbb{Q}^T\mathbb{Q} = \mathbb{1}$) and proper (i.e. $\det \mathbb{Q} = 1$). For this exercise $n = 2$ and the matrix Q that represents $\mathbb{Q}$ is 2×2:

$$Q = \begin{bmatrix} \vec{e}_1 \cdot \mathbb{Q}\vec{e}_1 & \vec{e}_1 \cdot \mathbb{Q}\vec{e}_2 \\ \vec{e}_2 \cdot \mathbb{Q}\vec{e}_1 & \vec{e}_2 \cdot \mathbb{Q}\vec{e}_2 \end{bmatrix} \equiv \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}.$$

(Here $\vec{e}_1 = \vec{e}_x$ and $\vec{e}_2 = \vec{e}_y$ are the standard Cartesian basis for $\mathbb{E}^2$.) Since $Q^TQ = I$ (cf. exercise 2.16(a)) is equivalent to the scalar equations $Q_{11}^2 + Q_{21}^2 = 1 = Q_{12}^2 + Q_{22}^2$ and $Q_{11}Q_{12} + Q_{21}Q_{22} = 0$, it follows that the columns of Q are mutually perpendicular vectors of unit length in $\mathbb{E}^2$. Therefore, there exists an angle α between 0 and 2π such that

$$\begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix} = \begin{bmatrix} \cos(\alpha \pm \pi/2) \\ \sin(\alpha \pm \pi/2) \end{bmatrix} = \begin{bmatrix} \mp \sin(\alpha) \\ \pm \cos(\alpha) \end{bmatrix}.$$

The requirement that $\det Q = 1$ removes the ambiguity in sign; we must have that the second column of Q is

$$\begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix} = \begin{bmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{bmatrix}.$$

Thus $Q = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ for some α between 0 and 2π.

The given information that $Q\vec{e}_x = \cos(\theta)\vec{e}_x + \sin(\theta)\vec{e}_y$ is equivalent to the matrix equation

$$Q\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}.$$

But

$$Q\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix},$$

so by inspection we may take $\alpha = \theta$. That is,

$$\boxed{Q = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}}.$$