Math 5222

Homework Set #3

#3.2 A ball-bearing of mass m is shot into the air vertically from a spring-loaded cannon whose muzzle is flush with the ground. The spring is linear with spring-constant k and is retracted a distance D from the muzzle. If the spring imparts all of its stored energy to the ball-bearing, how high does the ball-bearing fly? Neglect air drag and take the force of gravity to be constant.

The stored energy of the spring when fully retracted is

$$\int_0^D \vec{F}\cdot d\vec{x} = \int_0^D kx\,dx = \frac{kx^2}{2}\Big|_0^D = \frac{kD^2}{2}.$$

Note that the force $\vec{f}$ exerted on the ball-bearing is conservative, since

$$\vec{f}\cdot\dot{\vec{x}} = mg\dot{x} = (mgx)^{\cdot}$$

where $x(t) = \int_0^t \dot{x}(\tau)\,d\tau$ is the height reached by the object at t.

Therefore, the energy of the ball-bearing is conserved:

$$\frac{1}{2}mv^2 + mgx = K.E. + P.E. = \text{constant} = \frac{kD^2}{2}.$$

(since all of the stored energy of the spring is imparted to the ball-bearing)

At the peak of its flight, the ball-bearing has zero velocity so its kinetic energy is zero. Thus the maximum height $x = h$ of the ball-bearing satisfies

$$mgh = \frac{kD^2}{2} \quad \Rightarrow \quad \boxed{h = \frac{kD^2}{2mg}} .$$

#3.3 If $\vec{x} = t\vec{e}_x + t^2\vec{e}_y + t^3\vec{e}_z$ for $-\infty < t < \infty$, find (a) $\dot{\vec{x}}$; (b) $\ddot{\vec{x}}$.

(a) $\dot{\vec{x}} = \frac{d}{dt}\left(t\vec{e}_x + t^2\vec{e}_y + t^3\vec{e}_z\right) = \vec{e}_x + 2t\vec{e}_y + 3t^2\vec{e}_z$.

(b) $\ddot{\vec{x}} = \frac{d}{dt}(\dot{\vec{x}}) = 2\vec{e}_y + 6t\vec{e}_z$.

#3.4 Let $\vec{x} = \hat{x}(t)$ for $\alpha \le t \le \beta$ be the parametric representation of a curve C. If $\vec{x}$ is differentiable, the velocity $\vec{v} = \dot{\vec{x}}$ may, in view of (3.12) $ds/dt = |\hat{v}(t)|$, $s(t_1) = 0$, be represented in the form

$$\vec{v} = |\vec{v}|\mathbb{T} = \dot{s}\mathbb{T}. \tag{3.97}$$

$\mathbb{T}$ is called the <u>unit tangent to C</u>. If $\dot{s}$ and $\mathbb{T}$ are differentiable functions of t, then

$$\ddot{\vec{x}} = \dot{\vec{v}} = \vec{a} = \ddot{s}\mathbb{T} + \dot{s}\dot{\mathbb{T}} \equiv \vec{a}_p + \vec{a}_c. \tag{3.98}$$

$\vec{a}_p$ is called the <u>path acceleration</u> and $\vec{a}_c$ the <u>centripetal acceleration</u>. Note that if we differentiate both sides of the identity $\mathbb{T}\cdot\mathbb{T} = 1$, we obtain $\dot{\mathbb{T}}\cdot\mathbb{T} + \mathbb{T}\cdot\dot{\mathbb{T}} = 0$. Thus $\mathbb{T}\cdot\dot{\mathbb{T}} = 0$, which implies that $\vec{a}_p$ and $\vec{a}_c$ are orthogonal.

#3.4 (a) Show that

$$\vec{a}_p = \left(\frac{\ddot{\vec{x}}\cdot\dot{\vec{x}}}{\dot{\vec{x}}\cdot\dot{\vec{x}}}\right)\dot{\vec{x}}, \quad \vec{a}_c = \frac{\dot{\vec{x}}\times(\ddot{\vec{x}}\times\dot{\vec{x}})}{\dot{\vec{x}}\cdot\dot{\vec{x}}}. \tag{3.99}$$

Note that $\mathbb{T} = \dfrac{\dot{\vec{x}}}{|\dot{\vec{x}}|} = \dfrac{\dot{\vec{x}}}{\dot{s}}$, so $\vec{a}_p = \ddot{s}\,\mathbb{T} = \dfrac{\ddot{s}\dot{\vec{x}}}{\dot{s}} = \dfrac{\ddot{s}\dot{s}\dot{\vec{x}}}{(\dot{s})^2}$.

But $\sqrt{\dot{\vec{x}}\cdot\dot{\vec{x}}} = |\dot{\vec{x}}| = \dot{s}$ implies $\dot{\vec{x}}\cdot\dot{\vec{x}} = (\dot{s})^2$ so differentiating yields $\ddot{\vec{x}}\cdot\dot{\vec{x}} + \dot{\vec{x}}\cdot\ddot{\vec{x}} = 2\dot{s}\ddot{s}$ and hence $\ddot{\vec{x}}\cdot\dot{\vec{x}} = \ddot{s}\dot{s}$. Thus

$$\vec{a}_p = \left(\frac{\ddot{\vec{x}}\cdot\dot{\vec{x}}}{\dot{\vec{x}}\cdot\dot{\vec{x}}}\right)\dot{\vec{x}}.$$

Observe that $\mathbb{T} = \dfrac{\dot{\vec{x}}}{\dot{s}}$ so $\dot{\mathbb{T}} = (-1)(\dot{s})^{-2}\ddot{s}\dot{\vec{x}} + (\dot{s})^{-1}\ddot{\vec{x}} = (\dot{s})^{-2}[\dot{s}\ddot{\vec{x}} - \ddot{s}\dot{\vec{x}}]$

$= \dfrac{\dot{s}\ddot{\vec{x}} - \ddot{s}\dot{\vec{x}}}{\dot{\vec{x}}\cdot\dot{\vec{x}}}$. Therefore $\vec{a}_c = \dot{s}\dot{\mathbb{T}} = \dfrac{(\dot{s})^2\ddot{\vec{x}} - \dot{s}\ddot{s}\dot{\vec{x}}}{\dot{\vec{x}}\cdot\dot{\vec{x}}} = \dfrac{(\dot{\vec{x}}\cdot\dot{\vec{x}})\ddot{\vec{x}} - (\dot{\vec{x}}\cdot\ddot{\vec{x}})\dot{\vec{x}}}{\dot{\vec{x}}\cdot\dot{\vec{x}}}$

Applying the vector triple product identity

(1.22) $$(\vec{u}\times\vec{v})\times\vec{w} = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{v}\cdot\vec{w})\vec{u}$$

with $\vec{u} = \dot{\vec{x}}$, $\vec{v} = \ddot{\vec{x}}$, and $\vec{w} = \dot{\vec{x}}$ yields

$$\vec{a}_c = \frac{(\dot{\vec{x}}\times\ddot{\vec{x}})\times\dot{\vec{x}}}{\dot{\vec{x}}\cdot\dot{\vec{x}}} = \frac{\dot{\vec{x}}\times(\ddot{\vec{x}}\times\dot{\vec{x}})}{\dot{\vec{x}}\cdot\dot{\vec{x}}} .$$

#3.4 (b) Compute $\vec{a}_p$ and $\vec{a}_c$ for the curve

$$\vec{x} = t\vec{e}_x + t^2\vec{e}_y + t^3\vec{e}_z \qquad (-\infty < t < \infty)$$

in Exercise 3.3

From Exercise 3.3,

$$\ddot{\vec{x}}\cdot\dot{\vec{x}} = \left(\vec{e}_x + 2t\vec{e}_y + 3t^2\vec{e}_z\right)\cdot\left(2\vec{e}_y + 6t\vec{e}_z\right) = 4t + 18t^3$$

and $$\dot{\vec{x}}\cdot\dot{\vec{x}} = \left(\vec{e}_x + 2t\vec{e}_y + 3t^2\vec{e}_z\right)\cdot\left(\vec{e}_x + 2t\vec{e}_y + 3t^2\vec{e}_z\right) = 1 + 4t^2 + 9t^4.$$

$$\therefore\ \vec{a}_p = \left(\frac{\ddot{\vec{x}}\cdot\dot{\vec{x}}}{\dot{\vec{x}}\cdot\dot{\vec{x}}}\right)\dot{\vec{x}} = \boxed{\left(\frac{4t+18t^3}{1+4t^2+9t^4}\right)\left(\vec{e}_x + 2t\vec{e}_y + 3t^2\vec{e}_z\right)} .$$

$$\ddot{\vec{x}}\times\dot{\vec{x}} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 0 & 2 & 6t \\ 1 & 2t & 3t^2 \end{vmatrix} = -6t^2\vec{e}_x + 6t\vec{e}_y - 2\vec{e}_z$$

$$\Rightarrow\ \dot{\vec{x}}\times(\ddot{\vec{x}}\times\dot{\vec{x}}) = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 1 & 2t & 3t^2 \\ -6t^2 & 6t & -2 \end{vmatrix} = -(4t+18t^3)\vec{e}_x + (2-18t^4)\vec{e}_y + (6t+12t^3)\vec{e}_z$$

$$\therefore\ \vec{a}_c = \frac{\dot{\vec{x}}\times(\ddot{\vec{x}}\times\dot{\vec{x}})}{\dot{\vec{x}}\cdot\dot{\vec{x}}} = \boxed{\frac{-(4t+18t^3)\vec{e}_x + (2-18t^4)\vec{e}_y + (6t+12t^3)\vec{e}_z}{1+4t^2+9t^4}} .$$

#3.4 (c) Using the chain rule, show that

$$\vec{a}_c = \kappa \dot{s}^2 \mathbb{N}$$

and, from (a), that

$$\kappa = \frac{|\dot{\vec{x}} \times \ddot{\vec{x}}|}{|\dot{\vec{x}}|^3}. \qquad (3.101)$$

The definition of the centripetal acceleration is $\vec{a}_c = \dot{s}\dot{\mathbb{T}}$. But $\dot{\mathbb{T}} = \frac{d\mathbb{T}}{dt} = \frac{d\mathbb{T}}{ds}\cdot\frac{ds}{dt} = \mathbb{T}'\dot{s}$ by the chain rule so substituting and using the first Frenet formula (3.100): $\mathbb{T}' = \kappa \mathbb{N}$ gives the desired result.

$$\vec{a}_c = \dot{s}\left(\mathbb{T}'\dot{s}\right) = \dot{s}^2\mathbb{T}' = \dot{s}^2(\kappa\mathbb{N}) = \kappa\dot{s}^2\mathbb{N}.$$

Therefore

$$\vec{a}_c\cdot\vec{a}_c = (\dot{s}^2\kappa\mathbb{N})\cdot(\dot{s}^2\kappa\mathbb{N}) = (\dot{s})^4\kappa^2\,\mathbb{N}\cdot\mathbb{N} = (\dot{s})^4\kappa^2 = (\dot{\vec{x}}\cdot\dot{\vec{x}})^2\kappa^2.$$

Also part (a) implies

$$\vec{a}_c\cdot\vec{a}_c = \left(\left(\frac{\ddot{\vec{x}}\cdot\dot{\vec{x}}}{\dot{\vec{x}}\cdot\dot{\vec{x}}}\right)\dot{\vec{x}}\right)\cdot\left(\left(\frac{\ddot{\vec{x}}\cdot\dot{\vec{x}}}{\dot{\vec{x}}\cdot\dot{\vec{x}}}\right)\dot{\vec{x}}\right) = \left(\frac{\ddot{\vec{x}}\cdot\dot{\vec{x}}}{\dot{\vec{x}}\cdot\dot{\vec{x}}}\right)^2\dot{\vec{x}}\cdot\dot{\vec{x}} = \frac{(\ddot{\vec{x}}\cdot\dot{\vec{x}})^2}{\dot{\vec{x}}\cdot\dot{\vec{x}}}.$$

Equating the two expressions for $\vec{a}_c\cdot\vec{a}_c$ leads to

$$(\dot{\vec{x}}\cdot\dot{\vec{x}})^2\kappa^2 = \frac{(\ddot{\vec{x}}\cdot\dot{\vec{x}})^2}{\dot{\vec{x}}\cdot\dot{\vec{x}}} \quad \text{so} \quad \kappa^2 = \frac{(\ddot{\vec{x}}\cdot\dot{\vec{x}})^2}{(\dot{\vec{x}}\cdot\dot{\vec{x}})^3}.$$

But κ is clearly nonnegative by definition so

$$\kappa = |\kappa| = \sqrt{\kappa^2} = \sqrt{\frac{(\ddot{\vec{x}}\cdot\dot{\vec{x}})^2}{(\dot{\vec{x}}\cdot\dot{\vec{x}})^3}} = \frac{|\ddot{\vec{x}}\cdot\dot{\vec{x}}|}{|\dot{\vec{x}}|^3}.$$

#3.4 (d) Compute κ for the curve C: $\vec{x} = t\hat{e}_x + t^2\hat{e}_y + t^3\hat{e}_z$, $(-\infty < t < \infty)$, in Exercise 3.3.

From part (c) we have

$$\kappa = \frac{|\ddot{\vec{x}} \cdot \dot{\vec{x}}|}{|\dot{\vec{x}}|^3} = \frac{|(2\hat{e}_y + 6t\hat{e}_z)\cdot(\hat{e}_x + 2t\hat{e}_y + 3t^2\hat{e}_z)|}{|\hat{e}_x + 2t\hat{e}_y + 3t^2\hat{e}_z|^3} = \frac{|4t + 18t^3|}{\left(\sqrt{1+4t^2+9t^4}\right)^3}.$$

#3.4 (e) The unit vector $\mathbb{B} = \mathbb{T} \times \mathbb{N}$ is called the binormal to C. Show that $\mathbb{B}' = d\mathbb{B}/ds$ and $\mathbb{N}' = d\mathbb{N}/ds$ may be expressed in the form

(3.102) $$\mathbb{B}' = -\tau \mathbb{N}$$

(3.103) $$\mathbb{N}' = -\kappa \mathbb{T} + \tau \mathbb{B}.$$

The scalar function τ is called the torsion of C.

We derived the Frenet formulas

(3.100) $$\mathbb{T}' = \kappa \mathbb{N}$$

and (3.102)-(3.103) in lecture.

#3.4 (f) *Show that*

$$\tau = \frac{(\dot{\vec{x}} \times \ddot{\vec{x}}) \cdot \dddot{\vec{x}}}{|\dot{\vec{x}} \times \ddot{\vec{x}}|^2} .$$

From (3.102) and the definition of $\mathbb{B}$ we have

$$\tau = (-\tau \mathbb{N}) \cdot (-\mathbb{N}) = \mathbb{B}' \cdot (-\mathbb{N}) = -(\mathbb{T} \times \mathbb{N})' \cdot \mathbb{N}$$

$$= -(\mathbb{T}' \times \mathbb{N} + \mathbb{T} \times \mathbb{N}') \cdot \mathbb{N} = -\left[(\mathbb{T}' \times \mathbb{N}) \cdot \mathbb{N} + (\mathbb{T} \times \mathbb{N}') \cdot \mathbb{N}\right]$$

$$= -(\mathbb{T} \times \mathbb{N}') \cdot \mathbb{N} .$$

But the chain rule implies $\dot{\mathbb{N}} = \frac{d\mathbb{N}}{dt} = \frac{d\mathbb{N}}{ds}\frac{ds}{dt} = \dot{s}\,\mathbb{N}'$, so substituting in the above identity for τ gives

$$\tau = -\left(\mathbb{T} \times \frac{\dot{\mathbb{N}}}{\dot{s}}\right) \cdot N = -\frac{1}{\dot{s}}(\mathbb{T} \times \dot{\mathbb{N}}) \cdot \mathbb{N} .$$

We will now express $\dot{\vec{x}}$, $\ddot{\vec{x}}$, and $\dddot{\vec{x}}$ in terms of the Frenet frame: $\mathbb{T}$, $\mathbb{N}$, and $\mathbb{B}$. From

(3.97) $\qquad \dot{\vec{x}} = \dot{s}\,\mathbb{T}$

we have

$$(*) \qquad \ddot{\vec{x}} = \ddot{s}\,\mathbb{T} + \dot{s}\,\dot{\mathbb{T}} = \ddot{s}\,\mathbb{T} + \dot{s}\,|\dot{\mathbb{T}}|\left(\frac{\dot{\mathbb{T}}}{|\dot{\mathbb{T}}|}\right)$$

By definition, $\mathbb{N} = \frac{\mathbb{T}'}{|\mathbb{T}'|}$ and by the chain rule

$$\dot{\mathbb{T}} = \frac{d\mathbb{T}}{dt} = \frac{d\mathbb{T}}{ds}\frac{ds}{dt} = \dot{s}\,\mathbb{T}',$$

so $$\frac{\dot{\mathbb{T}}}{|\dot{\mathbb{T}}|} = \frac{\frac{d\mathbb{T}}{dt}}{\left|\frac{d\mathbb{T}}{dt}\right|} = \frac{\dot{s}\mathbb{T}'}{|\dot{s}\mathbb{T}'|} = \frac{\mathbb{T}'}{|\mathbb{T}'|} = \mathbb{N} .$$

Substituting in (*) gives

$$(**) \qquad \ddot{\vec{x}} = \ddot{s}\mathbb{T} + \dot{s}|\dot{\mathbb{T}}|\mathbb{N} \equiv \ddot{s}\mathbb{T} + \lambda\mathbb{N}$$

where to save space we have used the notation $\lambda = \dot{s}|\dot{\mathbb{T}}|$.

Differentiating (**) with respect to t (and assuming the curve is smooth enough to possess third derivatives at each point) yields

$$(***) \qquad \dddot{\vec{x}} = \dddot{s}\mathbb{T} + \ddot{s}\dot{\mathbb{T}} + \dot{\lambda}\mathbb{N} + \lambda\dot{\mathbb{N}} ,$$

or equivalently

$$(****) \qquad \frac{\dddot{\vec{x}} - \dddot{s}\mathbb{T} - \ddot{s}\dot{\mathbb{T}} - \dot{\lambda}\mathbb{N}}{\lambda} = \dot{\mathbb{N}} .$$

Substituting from (****) into the identity $\tau = -\frac{1}{\dot{s}}(\mathbb{T}\times\dot{\mathbb{N}})\cdot\mathbb{N}$ leads to

$$\tau = -\frac{1}{\dot{s}}\left[\mathbb{T}\times\left(\frac{\dddot{\vec{x}} - \dddot{s}\mathbb{T} - \ddot{s}\dot{\mathbb{T}} - \dot{\lambda}\mathbb{N}}{\lambda}\right)\right]\cdot\mathbb{N}$$

$$= -\frac{1}{\lambda\dot{s}}\left[(\mathbb{T}\times\dddot{\vec{x}})\cdot\mathbb{N} - \dddot{s}(\mathbb{T}\times\mathbb{T})\cdot\mathbb{N} - \ddot{s}(\mathbb{T}\times\dot{\mathbb{T}})\cdot\mathbb{N} - \dot{\lambda}(\mathbb{T}\times\mathbb{N})\cdot\mathbb{N}\right].$$

Clearly the second and fourth terms in square brackets are the zero vector. But $\dot{\mathbb{T}}$ is a scalar multiple of $\mathbb{N}$ (see the top line of this page, for example) and hence $(\mathbb{T}\times\dot{\mathbb{T}})\cdot\mathbb{N} = \vec{0}$ so

$$\tau = -\frac{1}{\lambda\dot{s}}(\mathbb{T}\times\dddot{\vec{x}})\cdot\mathbb{N}\ .$$

Using equations (**) and (3.97), it follows that

$$\tau = -\frac{1}{\lambda\dot{s}}\left(\mathbb{T}\times\dddot{\vec{x}}\right)\cdot\left(\frac{\ddot{\vec{x}}-\ddot{s}\mathbb{T}}{\lambda}\right)$$

$$= -\frac{1}{\lambda^2\dot{s}}\left[(\mathbb{T}\times\dddot{\vec{x}})\cdot\ddot{\vec{x}} - \ddot{s}(\mathbb{T}\times\dddot{\vec{x}})\cdot\mathbb{T}\right]$$

$$= -\frac{1}{\lambda^2\dot{s}}\left[(\mathbb{T}\times\dddot{\vec{x}})\cdot\ddot{\vec{x}}\right]$$

$$= -\frac{1}{\lambda^2\dot{s}}\left[\left(\frac{\dot{\vec{x}}}{\dot{s}}\right)\times\dddot{\vec{x}}\right]\cdot\ddot{\vec{x}}$$

$$= \frac{-1}{\lambda^2\dot{s}^2}\left(\dot{\vec{x}}\times\dddot{\vec{x}}\right)\cdot\ddot{\vec{x}}$$

$$= \frac{(\dot{\vec{x}}\times\ddot{\vec{x}})\cdot\dddot{\vec{x}}}{\lambda^2\dot{s}^2}\ .$$

But from equations (3.97) and (**) and the definition of $\mathbb{B}$ we have

$$\dot{\vec{x}}\times\ddot{\vec{x}} = (\dot{s}\mathbb{T})\times(\ddot{s}\mathbb{T}+\lambda\mathbb{N}) = \dot{s}\lambda(\mathbb{T}\times\mathbb{N}) = \dot{s}\lambda\mathbb{B}$$

so $|\dot{\vec{x}}\times\ddot{\vec{x}}|^2 = |\dot{s}\lambda\mathbb{B}|^2 = \dot{s}^2\lambda^2|\mathbb{B}|^2 = \dot{s}^2\lambda^2$. This yields the desired formula:

$$\tau = \frac{(\dot{\vec{x}}\times\ddot{\vec{x}})\cdot\dddot{\vec{x}}}{|\dot{\vec{x}}\times\ddot{\vec{x}}|^2}\ .$$

#3.4 (g) Compute τ for the curve $C: \vec{x} = t\hat{e}_x + t^2\hat{e}_y + t^3\hat{e}_z \quad (-\infty<t<\infty)$ in Exercise 3.3.

$$(\dot{\vec{x}} \times \ddot{\vec{x}}) \cdot \dddot{\vec{x}} = \begin{vmatrix} 1 & 2t & 3t^2 \\ 0 & 2 & 6t \\ 0 & 0 & 6 \end{vmatrix} = 12$$

$$\dot{\vec{x}} \times \ddot{\vec{x}} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\hat{e}_x - 6t\hat{e}_y + 2\hat{e}_z$$

$$|\dot{\vec{x}} \times \ddot{\vec{x}}|^2 = 36t^4 + 36t^2 + 4 .$$

$$\tau = \frac{(\dot{\vec{x}} \times \ddot{\vec{x}}) \cdot \dddot{\vec{x}}}{|\dot{\vec{x}} \times \ddot{\vec{x}}|^2} = \frac{12}{36t^4 + 36t^2 + 4} .$$