Problems from Math 5222 Lecture 1

Problems

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1. If we start with the definition of a vector $\mathbf{x}$ as an n-tuple of n real or complex numbers $(x_1, x_2, \ldots, x_n)$, and use for the definition of sum and product the formulas

$$\mathbf{x} + \mathbf{y}: \quad (x_1 + y_1, \ldots, x_n + y_n),$$
$$k\mathbf{x}: \quad (kx_1, \ldots, kx_n),$$
$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} \bar{x}_i y_i,$$

then

$$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z},$$
$$\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z},$$
$$(k\mathbf{x}) \cdot \mathbf{y} = \bar{k}(\mathbf{x} \cdot \mathbf{y}),$$
$$\mathbf{x} \cdot (k\mathbf{y}) = k(\mathbf{x} \cdot \mathbf{y}).$$

2. Prove that, if $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$ is a set of n linearly independent vectors in a complex n-dimensional vector space, then the only vector $\mathbf{x}$ orthogonal to each of the vectors $\mathbf{a}^{(i)}$ is the zero vector.

✓3. Prove that a set of mutually orthogonal nonzero vectors is always linearly independent.

✓4. Let the set of vectors $\mathbf{a}^{(i)}$ in E_n: $(a_1^{(i)}, a_2^{(i)}, \ldots, a_n^{(i)})$, $i = 1, 2, \ldots, n$, be linearly dependent, and suppose that r of them, $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(r)}$, $r < n$, are linearly independent. Show that every vector $\mathbf{x}$ that is orthogonal to this set of r linearly independent vectors forming the subset of E_n is also orthogonal to the remaining $n - r$ vectors in the given set.

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✓1. Write out in full the following expressions:

(a) $\delta_j{}^i a^j$. (b) $\delta_{ij} x^i x^j$. (c) $a_{ij} b_{jk} = \delta_{ik}$. (d) $a_{ijk} x^k$.

(e) $\dfrac{\partial f_i}{\partial x_j}\, dx_j$. (f) $\delta_i{}^i$. (g) $a^i = \dfrac{\partial x^i}{\partial y^j}\, b^j$. (h) $a_{ij(k)} x^j y^{(k)}$.

(i) $g_{ij} = \dfrac{\partial y^k}{\partial x^i}\dfrac{\partial y^k}{\partial x^j}$. (j) $a_{i(j)} x^{(j)}$. (k) $\delta_{ij}\delta^{jk}$.

The symbols $\delta_j{}^i$, δ_{ij}, and δ^{ij} all denote the Kronecker deltas.

✓2. Verify that (7.6) is the solution of (7.5).